A modest proposal for structuring public debt

by

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Bank of England
April 1997

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ABSTRACT

This paper proposes structuring public debt using considerations of robustness rather than strict optimality. Our proposal minimizes, over the infinite future, the conditional uncertainty surrounding public financing requirements. We estimate holding-period returns and market values on nominal and indexed UK government debt for a range of maturities, and derive the desired debt structures, according our proposal, that would be implied by the historical data. Although implications are not precise in all directions, given the historical UK data, our proposal leads to the government strongly favoring index-linked debt over conventionals.

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1. Introduction

The UK publishes each year a schedule of the debt it intends to issue the next financial year (see, e.g., HM Treasury, 1996). The schedule states the maturity composition of this debt and its division between nominal and index-linked. But what should inform such a decision on that schedule? This paper seeks to provide an analytical and empirical framework to address this question.

Public debt issuers everywhere confront similar financing problems. What economic findings have been available to guide them? Lucas and Stokey (1983) examined how the structure of government debt can be used to support optimal monetary and fiscal policies. Other relevant analyses are those of Barro (1995), Chari, Christiano, and Kehoe (1994), Marcet, Sargent, and Seppälä (1996), Missale (1995), Scott (1997), and Zhu (1992). The empirical analyses of Bohn (1990a, b) and Missale (1996) have considered how government debt can be structured to support a particular fiscal policy, namely martingale taxes (Barro, 1979).

This paper is similarly empirical, but it departs from previous work in two principal ways. The first innovation is that it puts forward a different objective for debt management. Rather than seek to support just one specific tax policy, our analysis proposes to take the tax process as given and instead to minimize the uncertainty surrounding it (or, more properly, surrounding the deficit, the excess of government outlays over tax revenues). Thus, our analysis is intended to apply, regardless of the precise fiscal policy.

The motivation for this derives, in part, from the stark disparity in the forms of optimal taxes across the studies of, among others, Barro (1979) on the one hand and Lucas and Stokey (1983) on the other, a disparity that has been emphasized in Marcet, Sargent, and Seppälä (1996), and Scott (1997). Since we see no consensus forming, we do not think it useful to propose debt structures to support one very special tax process rather than another.

Instead, we ask what features of optimal tax policies are common across a range of differing assumptions? What debt structures would be useful or sensible—not necessarily optimal—for all elements in that range? It is in this spirit of compromise that we offer what we call a *modest proposal* for structuring public debt. We present a model where optimal tax policy is determined but varies with alternative parametrizations of technology, preferences, and disturbances. Across parameterizations, the model restricts certain features of public debt but not others. Our proposal is simple: concentrate on those characteristics of public debt that are left unrestricted across optimal tax settings. By selecting features of the government debt structure according to our proposal, the decision maker never adversely affects the optimality of whatever fiscal policy is ultimately chosen, independent of whichever fiscal policy happens to be optimal for any particular economy.

Across the range of possibilities we consider, the implications that are common include the following.[1] (i) Neither optimal tax rates nor tax revenues are martingales. (ii) The implied deficit sequence varies systematically with disturbances to the economy: deficits are guaranteed to be neither smooth nor always zero. From the perspective of any given time period, future deficits have a predictable component that changes through time. (iii) The uncertainty around that predicted time path, however, is unrestricted. (iv) Given a tax policy, the total quantity of outstanding debt matters, but its composition is irrelevant.

Our specific proposal, then, is to use debt composition to reduce the conditional uncertainty surrounding the (optimal) time path predicted for government deficits. In the class of models we consider, this cannot reduce the desirability of a given fiscal policy. There is an aspect of robustness to this: If there were considerations—outside the model—that argued for greater predictability in government deficits, then our analysis provides that increased predictability without compromising the optimality of a given fiscal policy. If, on the other hand, such external considerations were absent or vacuous, our proposal does not harm: again, the optimality of a given fiscal policy is preserved. Below, we discuss, informally, what those external considerations might be.

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[1] As in most of the literature (e.g., Chari et al (1994), Marcet et al (1996), Scott (1997), and Zhu (1992)) we analyze only Ramsey policies.

It might seem peculiar to some readers that we use a model of debt structure *irrelevance* to make a proposal on the debt structure itself. Two factors drive our reasoning. First, such a framework clarifies the "robustness" properties of our proposal: our proposal respects all the criteria under which irrelevance obtains. If those criteria were weak, then this might be of little interest. However, in the current work, those criteria include, significantly, the optimality of dynamic tax policy. Second, such a framework is, we think, quite natural to describe an environment where one actor (the fiscal authority) selects tax rates and thus determines the primary deficit, before handing to another actor (the monetary authority) the decision on how to finance that deficit using different debt instruments. Such a scenario describes exactly the circumstances in many economies, including of course the UK.

While Section 2 and an Appendix below formalize the statements just made, we do not see that as the principal putative contribution of this paper. We believe that lies instead in the other innovation in our work where we extend previous empirical research on optimal public debt by taking into account *maturity* structures—on top of nominal versus index-linked or domestic-denominated versus foreign-denominated—to implement our proposal on reducing deficit uncertainty.[2]

The remainder of the paper is organized as follows. Section 2 describes the principal features of optimal dynamic tax policy, drawing out the implications already described above. Section 3 gives our specific proposal and relates it to previous work. Section 4 describes the empirical analysis motivated by our proposal. Section 5 discusses issues raised by our data construction; part of the value added in this paper is the putting together of a rich data set for further empirical analysis, both by ourselves and others. Section 6 then presents the empirical results following the analysis of Section 4. Section 7 concludes.

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[2] Of course, a central part of the theoretical analysis in Lucas and Stokey (1983) concerned precisely maturity structures for ensuring the time consistency of optimal policy. Although we discuss time consistency below, in our work its relation to maturity structures differs from that in Lucas and Stokey's.

An appendix contains proofs for the discussion in Section 2, and a Technical Appendix discusses econometric issues raised in Section 4.

2. Optimal tax policy

Our model for optimal tax policy excludes money and capital goods, and follows Lucas and Stokey (1983) in its broad outline. The model differs from Lucas and Stokey's in the following ways: First, we describe the stochastic environment explicitly as a time-stationary Markov chain with underlying state space including more than just government spending. This allows more explicit calculation of optimal tax policies but at the same time richer predictions on possible outcomes. The time stationarity reflects our interest in "regular" circumstances, rather than the wars and other extraordinary expenditures that Lucas and Stokey analyze (although their framework certainly includes time stationary as a special case; see, e.g., their Examples 8 and 9).

Extending the state space as we have done here also speaks to a small potential ambiguity in earlier work: does government spending matter for tax dynamics because it is directly part of the government's balance sheet constraint, or because it constitutes the underlying state of the economy? The observable implications differ: if the latter holds, then—as in Lucas and Stokey's analysis—tax rates have the same dynamics as government spending. If, by contrast, the former holds, then tax rates and government spending—while both functions of the same underlying state vector—need have no obvious dynamic relation between them. In either case, of course, the normative implications remain unchanged although the positive predictions differ.

Second, we allow the production technology to be stochastic and possibly nonlinear in labor, although we still exclude physical capital. We can thus study the implications for optimal fiscal policy of varying the production technology (from, say, productivity disturbances). Finally, third, the labor market is not specified to be competitive, so that labor's wage payment need not equal its marginal product.[3]

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[3] These extensions from Lucas and Stokey's original analysis, while useful to

Unlike Marcet et al (1996) and Scott (1997), we retain the assumption of complete markets in contingency claims. Together with the assumptions above, this captures one common intuition about a difference between financial and labor markets. Complete markets, however, is an important restriction, and we conjecture below how relaxing it would alter our conclusions.

We now briefly sketch our model, emphasizing where it differs from that in Lucas and Stokey (1983). The underlying state of the economy evolves in discrete time as $\omega = \{\omega_t : t = 0, 1, 2, \ldots\}$, where for each t the state is discrete:

$$\omega_t \in \Omega = \{\omega^{(1)}, \omega^{(2)}, \omega^{(3)}, \ldots\}.$$

From the perspective of time period 0, an *event* at time t is a history

$$\omega_0^t = (\omega_0, \omega_1, \ldots, \omega_t) \in \Omega \times \Omega \times \ldots \times \Omega.$$

Contingency claims at 0 on events at time t are, therefore, defined on a space that grows exponentially with t. Denote partial histories in the natural way

$$\forall s \geq t : \qquad \omega_t^s = (\omega_t, \omega_{t+1}, \ldots, \omega_s).$$

All actions and prices in the economy are functions of the possible histories ω_0^t.

Assume ω is a time-stationary Markov chain, with ω_0 given and for $t \geq 0$ transition probability $\Pr(\omega_{t+1} \mid \omega_0^t)$ equal to $\Pr(\omega_{t+1} \mid \omega_t)$ and invariant in t. This implies probabilities for all events of interest as:

$$\forall t,\ s \geq t+1 : \quad \Pr\left(\omega_{t+1}^s \mid \omega_0^t\right) = \Pr\left(\omega_{t+1}^s \mid \omega_t\right) = \prod_{r=1}^{s-t} \Pr(\omega_{t+r} \mid \omega_{t+r-1}).$$

A contingency claim on the event ω_0^t is an asset that pays 1 unit of the single nondurable consumption good at time t if ω_0^t occurs and 0 otherwise. Asset markets

make explicit, are trivial to perform—and the principal insights and conclusions remain unchanged.

are complete at time 0 when trade can occur in contingency claims on all possible events ω_0^t, for all $t \geq 1$.

The economy contains a single infinitely-lived representative (private) agent who values consumption c and leisure x. At time 0 given ω_0 the agent's utility is

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t(\omega_0^t), x_t(\omega_0^t)) = \sum_{t=0}^{\infty} \sum_{\omega_0^t \text{ given } \omega_0} \beta^t U(c_t(\omega_0^t), x_t(\omega_0^t)) \Pr(\omega_0^t \mid \omega_0), \quad (1)$$

with discount factor β between 0 and 1, and U strictly increasing and strictly concave jointly in its arguments. At time 0 the agent's endowment is 1 unit of leisure in each period $s \geq 0$, and a portfolio b_{-1} of public debt where

$$b_{-1} = \{b_{-1,s}, \ s = 0, 1, 2, \dots\},$$

with each $b_{-1,s}$ denoting quantities of contingent claims on time s events, i.e.,

$$b_{-1,s} = \{b_{-1,s}(\omega_0^s), \ \text{all } \omega_0^s\}.$$

Since the economy contains only a single representative agent, no private debt is ever issued. Notice that $b_{-1,s}$ for $s \geq 1$ is precisely long-term public debt. Analogously define

$$b_t = \{b_{t,s}, s \geq t+1\} = \{b_{t,s} = b_{t,s}(\omega_0^s), \ \text{all } s \geq t, \ \text{all } \omega_0^s\}$$

to be the agent's portfolio held at the end of period t.

When the agent supplies $(1 - x) \in [0, 1]$ units of time in work at time t, he receives $W_t(x, \omega_0^t)$ consumption goods as pre-tax labor income at that time (with $\partial W_t / \partial x < 0$). The amount W might be the outcome from the agent's own production technology or from his participating in a labor market. The notation says that given x, labor income can vary with random events, possibly from productivity disturbances.

In period t the government issues and retires debt, consumes $g_t(\omega_0^t)$, and levies proportional taxes on labor income $\tau_t(\omega_0^t) \in [0,1]$. While g is exogenously given, the government needs to select debt and taxes.

If at time t the agent works $1-x$, denote the implied total output in the economy by $Y_t(x, \omega_0^t)$, measured in units of the single nondurable consumption good. If the labor market is competitive then $W/(1-x)$ equals the partial derivative of Y with respect to x; in general, however, W and Y need not be thus related. Total resources in the economy bound the sum of private and government consumption

$$c_t(\omega_0^t) + g_t(\omega_0^t) = Y_t(x_t(\omega_0^t), \omega_0^t) \quad \forall t,\ \forall \omega_0^t. \tag{2}$$

Call p_0 the price of a unit of consumption good in period 0 and $p_t(\omega_0^t)$ the price at time 0 of a unit of consumption good in period t if event ω_0^t occurs. Assume that at time 0 the private agent takes prices p and tax rates τ as given, and formulates a *plan* for consumption, leisure, and asset portfolio holdings depending on potential histories, i.e., the consumer chooses for each $t \geq 0$ functions $c_t(\omega_0^t)$ and $x_t(\omega_0^t)$, and the set of functions b_t.

In period t if ω_0^t has occurred, the consumer faces the budget constraint (stated in period 0 prices):

$$\begin{aligned} p_t(\omega_0^t)c_t(\omega_0^t) + \sum_{s=t+1}^{\infty} \sum_{\omega_{t+1}^s|\omega_0^t} p_s(\omega_0^s) b_{t,s}(\omega_0^s) \\ = p_t(\omega_0^t)[(1-\tau_t(\omega_0^t))W_t(x_t(\omega_0^t), \omega_0^t) + b_{t-1,t}(\omega_0^t)] \\ + \sum_{s=t+1}^{\infty} \sum_{\omega_{t+1}^s|\omega_0^t} p_s(\omega_0^s) b_{t-1,s}(\omega_0^s). \end{aligned} \tag{3}$$

Given ω_0^t, all the histories ω_0^s that do not include ω_0^t are, of course, impossible and so contingency claims on them are worthless. The notation takes this into account by writing $\omega_{t+1}^s \mid \omega_0^t$ to denote only those histories ω_0^s that are continuations of ω_0^t.

The first term on the right of (3) is the sum of post-tax labor income and revenue from the only contingency claim that pays off in event ω_0^t. The second term is the total value of all the other still unredeemed contingency claims carried over from the previous period. The right side therefore shows the total resources controlled by the consumer in period t when event ω_0^t has occurred. The left side shows total disbursements: the first term is the value of consumption; the second is the total value of the portfolio to be carried into the following period.

While (3) contains summations over conditional events $\omega_{t+1}^s \mid \omega_0^t$, the reader should remember that (3) is a constraint that holds for each event ω_0^t. In particular, (3) is not an expectational equation—the summation over events ω_0^t does not involve probabilities, but instead is a summation across different assets.

The Appendix verifies that maximizing (1) subject to (3) for all t and all ω_0^t gives first-order conditions:

$$\frac{U_x(\omega_0^t)}{U_c(\omega_0^t)} = -\left(1 - \tau_t(\omega_0^t)\right) W_x(\omega_0^t) \qquad \forall t \geq 0,\ \forall \omega_0^t \tag{4}$$

and

$$\beta^t \frac{U_c(\omega_0^t)}{U_c(\omega_0)} \Pr(\omega_1^t \mid \omega_0) = \frac{p_t(\omega_0^t)}{p_0} \qquad \forall t \geq 1,\ \forall \omega_0^t, \tag{5}$$

where $U_x(\omega_0^t)$ and $U_c(\omega_0^t)$ denote $\partial U/\partial x$ and $\partial U/\partial c$ evaluated at $(c_t(\omega_0^t), x_t(\omega_0^t))$, and $W_x(\omega_0^t)$ denotes $\partial W/\partial x$ evaluated at $(x_t(\omega_0^t), \omega_0^t)$.

Again, (4) and (5) are not expectational equations, but instead hold at each event ω_0^t. Since U is strictly concave, given (τ, p), equations (3), (4), and (5) admit a unique solution (c, x) for all $t \geq 0$ and all ω_0^t.

In time t the government faces the constraints:

$$\begin{aligned} p_t(\omega_0^t)\tau_t(\omega_0^t)W_t(x_t(\omega_0^t), \omega_0^t) &+ \sum_{s=1}^{\infty} \sum_{\omega_{t+1}^s \mid \omega_0^t} b_{t,s}(\omega_0^s) p_s(\omega_0^s) \\ &= p_t(\omega_0^t)\left[g_t(\omega_0^t) + b_{t-1,t}(\omega_0^t)\right] + \sum_{s=1}^{\infty} \sum_{\omega_{t+1}^s \mid \omega_0^t} p_s(\omega_0^s) b_{t-1,s}(\omega_0^s) \qquad \forall \omega_0^t \end{aligned} \tag{6}$$

The interpretation of this is similar to that of (3) for the consumer. The first summand on the right is the total payout from government expenditures and the one contingency claim redeemable in state ω_0^t; the second summand is the total value of the still outstanding public debt at the beginning of the period. The left side is the sum of tax receipts and the total value of public debt to be carried into the following period.

In the Appendix, we verify that the collection of constraints (6) for all $t \geq 0$ and all ω_0^t consistent with ω_0 is equivalent to the single constraint

$$p_0 \left[\tau_0 W_0(x_0) - g_0\right] + \sum_{s=1}^{\infty} \sum_{\omega_1^s | \omega_0} p_s(\omega_0^s) \left[\tau_s(\omega_0^s) W_s(x_s(\omega_0^s), \omega_0^s) - g_s(\omega_0^s)\right]$$
$$= p_0 b_{-1,0} + \sum_{s=1}^{\infty} \sum_{\omega_1^s | \omega_0} p_s(\omega_0^s) b_{-1,s}(\omega_0^s) \qquad (7)$$

Upon substituting in equilibrium prices p_s from (5) as functions of (c, x) and the transition probabilities $\Pr(\cdot \mid \cdot)$, equation (7) is sometimes called an *implementability constraint* (e.g., Chari et al (1994) and Marcet et al (1996)). Here, we note it is just a restatement of period 0 budget balance. The right hand side is the total value in period 0 of all public debt outstanding; the left hand side is the fully contingent planned excess of tax receipts over government expenditure.

The importance of the equivalence just drawn is the following: the public debt portolio b_{-1} matters for government policy only through the expression on the right of (7). Below, we will see that once tax policies are fixed, so will be prices p. But in that case, portfolio b_{-1} is important only through its total market value: its exact composition—provided all contingency claims markets remain open—is irrelevant.

Solve for τ from the first-order condition (4) to give:

$$\tau W = \left(1 + \frac{U_x}{U_c W_x}\right) W$$

Subtracting g from this and using the national income constraint (2), we get

$$\tau W - g = \left(1 + \frac{U_x}{U_c W_x}\right) W - (Y - c).$$

Next, using (5) to solve out p_s (all $s \geq 0$), equation (7) becomes

$$\begin{aligned} U_c(\omega_0) & \left[\left(1 + \frac{U_x(\omega_0)}{U_c(\omega_0) W_x(\omega_0)}\right) W_0(x_0, \omega_0) - Y_0 + c_0 \right] \\ & + \sum_{s=1}^{\infty} \sum_{\omega_1^s | \omega_0} \beta^s U_c(\omega_0^s) \Pr(\omega_1^s \mid \omega_0) \\ & \quad \times \left[\left(1 + \frac{U_x(\omega_0^s)}{U_c(\omega_0^s) W_x(\omega_0^s)}\right) W_s(x_s(\omega_0^s), \omega_0^s) - Y_s(\omega_0^s) + c_s(\omega_0^s) \right] \\ & = U_c(\omega_0) b_{-1,0} + \sum_{s=1}^{\infty} \sum_{\omega_1^s | \omega_0} U_c(\omega_0^s) \Pr(\omega_1^s \mid \omega_0) b_{-1,s}(\omega_0^s) \end{aligned}$$

Define the left and right sides of this equation to be ξ and ψ respectively, so that it becomes simply:

$$\xi(c, x; \omega) = \psi(b_{-1}, c, x; \omega). \tag{8}$$

We now complete the argument: the government is benevolent, and formulates a (Ramsey) plan to maximize the agent's utility (1) subject to the single constraint (8). The Lagrangean is

$$\begin{aligned} \mathfrak{L} = & \left[\sum_{t=0}^{\infty} \sum_{\omega_1^t | \omega_0} \beta^t U(c_t(\omega_1^t), x_t(\omega_0^t)) \Pr(\omega_1^t \mid \omega_0) \right] \\ & \qquad - \lambda \times \left(\xi(c, x; \omega) - \psi(b_{-1}, c, x; \omega) \right), \end{aligned}$$

so that, assuming an interior solution, the first-order conditions are for all t and

ω_0^t:

$$c_t(\omega_0^t): \qquad \beta^t U_c(\omega_0^t)\,\Pr(\omega_1^t \mid \omega_0) - \lambda \times \left(\frac{\partial \xi}{\partial c_t(\omega_0^t)} - \frac{\partial \psi}{\partial c_t(\omega_0^t)}\right) = 0$$

$$x_t(\omega_0^t): \qquad \beta^t U_x(\omega_0^t)\,\Pr(\omega_1^t \mid \omega_0) - \lambda \times \left(\frac{\partial \xi}{\partial x_t(\omega_0^t)} - \frac{\partial \psi}{\partial x_t(\omega_0^t)}\right) = 0$$

Studying these equations, it is apparent that if output Y, labor income W, and government spending g are time-stationary in ω, i.e.,

$$Y_t(x, \omega_0^t) = Y(x, \omega_t), \qquad W_t(x, \omega_0^t) = W(x, \omega_t), \qquad g_t(\omega_0^t) = g(\omega_t),$$

then the solutions

$$c_t(\omega_0^t) = c(\omega_t) \quad \text{and} \quad x_t(\omega_0^t) = x(\omega_t)$$

will also have the same property. But then so will tax rates, total tax revenues, and the primary deficit

$$\begin{aligned} \tau_t(\omega_0^t) &= 1 + \frac{U_x(\omega_0^t)}{U_c(\omega_0^t) W_x(\omega_0^t)} = \tau(\omega_t), \\ \tau_t(\omega_0^t) W_t(\omega_0^t) &= \tau(\omega_t) W(x(\omega_t), \omega_t), \\ g_t(\omega_0^t) - \tau_t(\omega_0^t) W_t(\omega_0^t) &= g(\omega_t) - \tau(\omega_t) W(x(\omega_t), \omega_t), \end{aligned} \tag{9}$$

while the government deficit itself

$$g_t(\omega_0^t) - \tau_t(\omega_0^t) W_t(\omega_0^t) - b_{t-1,t}(\omega_0^t) \tag{10}$$

has dynamics that might be more intricate, depending on the structure of public debt b_{t-1}.

The time path of the primary deficit is predictable and varies systematically with the forecast horizon: the expected time path can be calculated from:

$$\begin{aligned} &E(g(\omega_s) - \tau(\omega_s) W(x(\omega_s), \omega_s) \mid \omega_0^t) \\ &\qquad = E(g(\omega_s) - \tau(\omega_s) W(x(\omega_s), \omega_s) \mid \omega_t) \qquad \forall s > t. \end{aligned}$$

Given τ, prices are determined, and then only the market value of b_{-1} in ψ matters:

$$\psi(b_{-1}, c, x; \omega) = p_0 b_{-1,0} + \sum_{s=1}^{\infty} \sum_{\omega_1^s | \omega_0} p_s(\omega_0^s) b_{-1,s}(\omega_0^s).$$

Thus, fixing the tax policy, the composition of public debt is irrelevant.[4] That composition, however, does matter for (10), and in particular the conditional variability of the government deficit will change with debt composition—even if the optimality properties remain invariant.

The sequence for optimal taxes depends on the parameters of Y, W, U, and the process ω, or in words, technology, preferences, and the disturbances to the economy. Varying the values of those parameters will then also alter the optimal tax policy.

What happens if markets are incomplete, as in Marcet et al (1996), or Scott (1997)? The range of possibilities remains to be fully worked out in the literature. However, we conjecture that while the precise form of the left side of equations (9) will change, their probability characteristics will remain Markov—indeed, specifying a Markov subgame perfect equilibrium in the more complicated situations studied in Marcet et al (1996) would make that automatic. Thus, we conjecture that our proposal below will continue to have robustness properties even in the incomplete markets case.

To summarize then we simply refer the reader back to statements (i)–(iv) in the previous Section: it is these that we use in the remainder of the paper. We do not exploit the precise form of the optimal tax policy as that will vary with parameters of the economy.

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[4] The calculations, of course, show that the portfolio composition *does* matter for determining tax policy: but once a tax policy is decided then the government debt portfolio matters only through its total market value.

3. Minimizing uncertainty around deficit paths

Previous empirical studies (e.g., Bohn (1990a, b) and Missale (1996)) consider arranging the public debt structure around a particular tax policy. We have just argued that we don't wish to do this.

Our proposal, instead, is to structure government debt to minimise unexpected variations in the expected time path of primary deficit. We think this consistent with the broad conclusions of the previous section. In our approach, the deficit itself is free to vary predictably as a function of the underlying disturbances to taxes, government spending, aggregate income, and other macroeconomic conditions: it is only *unexpected* variability that should be minimised.[5]

When unexpected variations in the budget deficit adversely affect fiscal credibility, our uncertainty-minimizing proposal can also be given a more positive justification, beyond just robustness. Rather than structure debt to support one purportedly optimal tax policy, the debt manager takes taxes as given and structures debt to best support the credibility of whatever fiscal policy is chosen. In this interpretation, predictable variations in the budget deficit due, say, to cyclical factors, do not adversely affect fiscal credibility. However, large unexpected variations do, as they cause private agents to reassess their perception of the authorities' fiscal rule.

We now make precise our notion of uncertainty minimizing. For simplicity, the discussion here considers only index-linked gilts. It is easily generalised to incorporate nominal gilts as well, complicating only the notation.

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[5] Our concern with variability implies that we ignore mean considerations: a gilt might need to be in an optimal portfolio because its average payout is low, even if it contributes greatly to unpredictability. We ignore this mean effect, but as should become clear below, had we not done so, our empirical results would simply be strengthened. Another shortcoming in our approach is that if policy regimes, both now and in the future, are known with certainty, better-informed optimal debt structures could be calculated—by explicitly defining the debt manager's objective using that information. But we think such knowledge unlikely.

Assume the timing convention where all trades, redemptions, and new issues occur at the end of the period. Let $P_{k,t}$ denote the period-t price, in consumption goods numeraire, of one unit of zero-coupon index-linked government debt maturing in period $k+t$. These gilts might currently be mid-stream, i.e., they might have been issued in an earlier period $t-s$ with then time-to-maturity $k+s$ for any positive s. However, since all such gilts have identical income streams, they must have the same price. Each gilt pays off at par on maturity. Thus, by choice of units, we can write:

$$P_{0,t} = 1 \quad \text{for all } t.$$

Let $D_{k,t}$ denote the par value outstanding in period t of all gilts maturing at $t+k$. Letting G_t denote government spending, and tax revenues T_t include seignorage from issuing base money, the government budget constraint at t can be written as:

$$\sum_{k\geq 1} (D_{k,t} - D_{k+1,t-1})\, P_{k,t} = G_t - T_t + D_{1,t-1}. \tag{11}$$

The right-hand side of this equation is the primary deficit $G_t - T_t$ together with the par value of debt maturing at t. The left-hand side is the total value, summed over all maturity horizons, of new debt issue evaluated at current market prices. For a given t, the change in par value $D_{k,t} - D_{k+1,t-1}$ will typically be zero for most k's. Note that equation (11) allows $D_{k,t} - D_{k+1,t-1}$ for any given k to be negative, indicating where the government retired debt.

Rearrange (11) as:

$$\begin{aligned}
\sum_{k\geq 1} P_{k,t} D_{k,t} &= G_t - T_t + D_{1,t-1} + \sum_{k\geq 1} P_{k,t} D_{k+1,t-1} \\
&= G_t - T_t + D_{1,t-1} + \sum_{k>1} P_{k-1,t} D_{k,t-1} \\
&= G_t - T_t + \sum_{k\geq 1} P_{k-1,t} D_{k,t-1},
\end{aligned}$$

using the par condition $P_{0,t} = 1$. Then, subtracting $\sum_{k\geq 1} P_{k,t-1}D_{k,t-1}$ from both sides gives:

$$\begin{aligned}
\sum_{k\geq 1}&(P_{k,t}D_{k,t} - P_{k,t-1}D_{k,t-1}) \\
&= G_t - T_t + \sum_{k\geq 1}\left(P_{k-1,t} - P_{k,t-1}\right)D_{k,t-1} \\
&= G_t - T_t + \sum_{k\geq 1}\left(\frac{P_{k-1,t} - P_{k,t-1}}{P_{k,t-1}}\right)P_{k,t-1}D_{k,t-1} \\
&= G_t - T_t + \sum_{k\geq 1} R_{k,t}\left(P_{k,t-1}D_{k,t-1}\right), \qquad (12)
\end{aligned}$$

where $R_{k,t}$ is the holding-period return over period t on gilts maturing in period $k + t - 1$.

Let $B_{k,t} \stackrel{\text{def}}{=} P_{k,t}D_{k,t}$: this is the real market value of gilts outstanding at the end of period t that will mature in period $t + k$. Define the vectors

$$\begin{aligned}
B_t &= (\, B_{1,t}, B_{2,t}, B_{3,t}, \dots\,)', \\
R_t &= (\, R_{1,t}, R_{2,t}, R_{3,t}, \dots\,)';
\end{aligned}$$

and letting $\mathbf{1}$ denote a vector of 1's, decompose B_t as follows:

$$B_t = b_t\widetilde{B}_t \quad \text{where } b_t \in \mathbb{R} \text{ and } \mathbf{1}'\widetilde{B}_t = 1,$$

so that b_t is the real market value of all outstanding government debt, and $\widetilde{B}_t$ is the vector of *portfolio shares*. Equation (12) can then be written:

$$\begin{aligned}
b_t - b_{t-1} &= G_t - T_t + R_t'\left(b_{t-1}\widetilde{B}_{t-1}\right) \\
&= G_t - T_t + \left(b_{t-1}R_t\right)'\widetilde{B}_{t-1}.
\end{aligned}$$

Define the left-hand side to be

$$d_t \stackrel{\text{def}}{=} b_t - b_{t-1};$$

this is the change over period t in the total market value of outstanding government debt. If positive, it is the total value of new debt issued; if negative, it is the total value of debt retired before maturity.

At the end of period t the policy maker or debt manager observes information set *info*(t), assumed to include period-t realisations of exogenous variables. For β a discount factor between 0 and 1 the policy maker seeks to minimise the conditional variance of the present discounted value of all future deficits. Thus, by uncertainty minimizing at time t we mean a sequence of *policy functions* for portfolio shares $\widetilde{B}_{t+j}$, one for each positive j, with each such policy function measurable in *info*$(t+j)$, so that the sequence solves:

$$\min_{\{\widetilde{B}_{t+j}:j\geq 0\}} \mathrm{Var}\left(\sum_{j\geq 0} \beta^j d_{t+j} \mid \mathit{info}(t)\right) \tag{13}$$

subject to (for all $j \geq 0$):

$$d_{t+j} = G_{t+j} - T_{t+j} + (b_{t+j-1} R_{t+j})' \, \widetilde{B}_{t+j-1}; \tag{14}$$

$$b_{t+j} = b_{t+j-1} + d_{t+j}; \tag{15}$$

$$\{\, G_{t+j}, T_{t+j}, R_{t+j} : j \geq 0 \,\} \text{ exogenous;} \tag{16}$$

$$b_{t-1},\ \widetilde{B}_{t-1} \text{ given.} \tag{17}$$

The sequence of events is as follows: at the beginning of period t the policy maker takes as given the historical values for b_{t-1} and $\widetilde{B}_{t-1}$. Over period t, uncertainty resolves in the form of particular realisations for G_t, T_t, and R_t, thereby determining the deficit d_t through the $j = 0$ version of equation (14). The total market value of debt outstanding b_t is then determined by equation (15).

At the end of period t, the policy maker formulates a *plan*:

$${}_t\widetilde{B} \stackrel{\text{def}}{=} \left\{ {}_t\widetilde{B}_{t+j}(\mathit{info}(t+j)) : j \geq 0 \right\}$$

where ${}_t\widetilde{B}_t(\mathit{info}(t))$ is implemented—thereby determining $\widetilde{B}_t$ as initial condition for the next period—while ${}_t\widetilde{B}_{t+j}(\mathit{info}(t+j))$, for $j \geq 1$, is a sequence of contingency actions. For each positive j, the function ${}_t\widetilde{B}_{t+j}$ is a contingency action optimal from the perspective of period t, taking value that varies with information that will arrive only at time $t + j$.

When we say that the holding-period returns R_t realise exogenously, what we mean is that policy-makers view market-clearing prices $\{P_{k-1,t} : k > 1\}$ as determined outside their control.[6] Those are the market prices at which gilts transactions occur at the end of period t. Thus, even if policy makers were to do nothing, $\widetilde{B}_t$ would typically differ from $\widetilde{B}_{t-1}$. In general, however, at the end of period t trade occurs to clear markets: in the course of those transactions the policy maker re-adjusts portfolio shares to arrive at the desired $\widetilde{B}_t$. That vector of portfolio shares obviously depends on G_t, T_t, and R_t, but it could also vary with other information in $\mathit{info}(t)$ helpful for prediction. The process then repeats.

The objective function (13) does not lead to the standard Bellman recursive formulation: the conditional variance operator, unlike conditional expectations, is not linear in its argument. Thus, standard dynamic programming arguments are unavailable for solving (13)–(17).

4. The empirical model

In light of these non-standard technical complications, we adopt the following approach. First, we restrict the class of policy functions to consider. This class will admit a closed-form analytical solution for the optimal plan ${}_t\widetilde{B}$. We then describe restrictions on the stochastic environment under which (the continuation of) that solution remains optimal—in the original class of policy functions—for all

[6] We take this as the standard assumption. A researcher might wish to let policy makers take into account the price effects of their quantity decisions by letting holding-period returns explicitly vary with quantities outstanding in (14). We have chosen not to do so here as it seemed to complicate matters without adding much insight.

future periods, i.e.,

$$ {}_t\widetilde{B}_{t+j} = {}_{t+j}\widetilde{B}_{t+j}, \quad \text{for all } j \geq 1. $$

The optimal plan in the class of restricted policy functions is then also time consistent.[7]

Say that a plan at t is *constant* when:

$$ {}_t\widetilde{B}_{t+j}(\mathit{info}(t+j)) = {}_t\widetilde{B}_t(\mathit{info}(t)), \qquad \text{for all } j \geq 1. $$

In words, in a constant plan, the debt manager intends to maintain portfolio shares at the value currently chosen.

For constant plans, the criterion (13) simplifies. To see this, first define the vector

$$ \widetilde{R}_t = b_{t-1}R_t. $$

The entries in $\widetilde{R}$ have a simple interpretation: each entry is the holding-period cost were the entire stock to be held in just that single gilt.

Using this in (14) gives

$$ d_{t+j} = G_{t+j} - T_{t+j} + \widetilde{R}_{t+j}\widetilde{B}_{t+j-1}. \tag{18} $$

Since R is taken as exogenous by the debt manager, so is $\widetilde{R}$, except in the degenerate case when $b = 0$, which we rule out.

Let the vector time-series process $\{W_t : \text{integer } t\}$ be that which induces the sequence of information sets $\mathit{info}(t)$, i.e., for each t, the information $\mathit{info}(t)$ is the sigma-algebra generated by the random variables in $\{W_s : s \leq t\}$. Observations on current and lagged W constitute market participants' information. The vector W_t includes at least $\left(\widetilde{R}'_t, G_t, T_t\right)'$ as its first entries.

[7] The concern with time consistency here differs from that in Lucas and Stokey (1983). We treat only the problem of the debt manager; that of private agents is nowhere made explicit. Nevertheless, a potential time inconsistency arises as the criterion function (13) does not have the usual Bellman recursive formulation.

Assume that conditional on *info*(t) the vector W_{t+j} has a linear innovations representation:

$$\begin{aligned} W_{t+j} \mid \mathit{info}(t) &= \sum_{k=0}^{j-1} c(k)\epsilon_{t+j-k}, \\ E\left(\epsilon_{t+k} \mid \mathit{info}(t)\right) &= 0 \text{ a.s. for all } k \geq 1. \end{aligned} \tag{19}$$

This would be implied, for instance, by $\{W_t\}$ being covariance stationary and linearly regular so that it then admitted a Wold representation. But such an assumption on W is only sufficient, not necessary. Representation (19) would remain even if W were not covariance stationary but instead integrated, or even cointegrated and having mixed covariance stationary and difference stationary elements. The latter cases are not conceptually subtle, but the manipulations to see them are given in the Technical Appendix: those computations are convenient to have all in one place and will be used for the empirical results to follow.

Next, from (19), define the matrix function

$$\Gamma(z) = \sum_{j=0}^{\infty} c(j)z^j, \quad |z| \leq 1.$$

We assume that

$$\|\Gamma(z)\| < \infty \quad \text{for all } |z| < 1;$$

this bound would hold automatically if W were covariance stationary. It is easily shown to continue to hold even if W were only difference stationary.

Define $\gamma(\widetilde{B})$ to be the vector having the same dimension as W such that in (18) we can write

$$d_{t+j} = \gamma(\widetilde{B}_{t+j-1})'W_{t+j}, \tag{20}$$

i.e., when W_t begins with $\left(\widetilde{R}_t', G_t, T_t\right)'$, then $\gamma(\widetilde{B})$ begins with $\left(\widetilde{B}', 1, -1\right)'$ and is thereafter 0.

Then for ${}_t\widetilde{B}$ a constant plan, the objective function (13) becomes

$$\operatorname{Var}\left(\sum_{j\geq 0} \beta^j d_{t+j} \;\middle|\; \mathit{info}(t) \right)$$

$$= \operatorname{Var}\left(\sum_{j\geq 1} \beta^j \gamma(\widetilde{B}_t)' W_{t+j} \;\middle|\; \mathit{info}(t) \right)$$

(since $j = 0$ carries no uncertainty)

$$= \operatorname{Var}\left(\gamma(\widetilde{B}_t)' \sum_{j\geq 1} \beta^j W_{t+j} \;\middle|\; \mathit{info}(t) \right)$$

(since $\gamma(\widetilde{B}_t)$ is invariant in j)

$$= \operatorname{Var}\left(\gamma(\widetilde{B}_t)' \sum_{j\geq 1} \left[\beta^j \sum_{k=0}^{j-1} c(k)\epsilon_{t+j-k} \right] \;\middle|\; \mathit{info}(t) \right)$$

(definition of conditional variance)

$$= \gamma(\widetilde{B}_t)' \left[\operatorname{VCV}\left(\sum_{k=1}^{\infty} \left[\sum_{j=0}^{\infty} c(j)\beta^j \right] \beta^k \epsilon_{t+k} \;\middle|\; \mathit{info}(t) \right) \right] \gamma(\widetilde{B}_t)$$

(interchanging orders of summation).

where VCV denotes variance-covariance matrix.

Using the definition of Γ, the final expression becomes

$$\operatorname{Var}\left(\sum_{j\geq 0} \beta^j d_{t+j} \;\middle|\; \mathit{info}(t) \right)$$

$$= \gamma(\widetilde{B}_t)' \Gamma(\beta) \operatorname{VCV}\left(\sum_{k=1}^{\infty} \beta^k \epsilon_{t+k} \;\middle|\; \mathit{info}(t) \right) \Gamma(\beta)' \gamma(\widetilde{B}_t)$$

Defining the conditional VCV

$$\Omega_t = \mathrm{VCV}\left(\sum_{k=1}^{\infty} \beta^k \epsilon_{t+k} \;\middle|\; \mathit{info}(t) \right),$$

the objective function then is

$$\mathrm{Var}\left(\sum_{j \geq 0} \beta^j d_{t+j} \;\middle|\; \mathit{info}(t) \right) = \gamma(\widetilde{B}_t)' \Gamma(\beta) \Omega_t \Gamma(\beta)' \gamma(\widetilde{B}_t) \tag{21}$$

The constant-plan uncertainty-minimizing gilts portfolio minimizes (21) subject to $\mathbf{1}'\widetilde{B}_t = 1$. Since time variation in (21) enters only through Ω_t, it is immediate that the optimal plan will be time consistent provided W is conditionally homoskedastic.[8] Then,

$$\Omega_t = \Omega_0 = \frac{\beta^2}{1-\beta^2} \mathrm{VCV}(\epsilon_0).$$

If, on the other hand, W is conditionally heteroskedastic, then the policy maker should, in general, seek to adjust $\widetilde{B}$ as a function of the changing covariance structure. We have not been able to characterize optimal plans in that case. In future research, obtaining such a more general solution would take high priority. For now, however, we will assume conditional homoskedasticity.

Equation (21) makes clear a number of properties for the optimal debt structure. First, the uncertainty-minimizing debt structure depends both on W's conditional variance Ω and on W's dynamics, i.e., the entire sequence of W's conditional

[8] That a constant plan and conditional homoskedasticity go together is intuitively plausible. Moreover, the development in the text makes apparent that without the latter, the former could not be optimal. Note, though, that without first restricting to constant plans, the objective function (13) is not time invariant even under conditional homoskedasticity. The relation between the two is thus more subtle than might at first appear.

means through time, Γ. That dependence is, in general, intricate, but abstracting from details, we can intuit a number of features. The uncertainty-minimizing portfolio will exploit covariances between returns on the one hand and government spending and taxes on the other. Gilts with low conditional-variance returns will be favoured except when doing so increases (21) through their positive conditional covariance with government spending or negative conditional covariance with tax receipts.

A different reason that gilts with low conditional-variance returns might not be favoured is if the impact on conditional means—through the propagation mechanism Γ—is long-lived, thereby contributing to long-run volatility. That long-run volatility, because it enters as $\Gamma(\beta)$, is directly influenced by the policy maker's discount factor β: the lower is β, the less important becomes the propagation mechanism, and the more important the instantaneous conditional volatility Ω.

In the time-consistent case, an optimal plan could, in principle, imply a negative entry in $\widetilde{B}$. This means that the government should accept debt issued by the public. Isolated negative entries in $\widetilde{B}$ are perfectly consistent with the government maintaining a positive total quantity of debt. Historically, however, governments have chosen only to issue debt, not to lend in net to the public. We will calculate optimal portfolio plans with and without non-negativity restrictions on $\widetilde{B}$: this then permits evaluating the cost of differing loan arrangments between the government and the public.

Thus, for $\Omega_t = \Omega$ constant, we consider the problem:

$$\min_{\widetilde{B}} \gamma(\widetilde{B})'\Gamma(\beta)\Omega\Gamma(\beta)'\gamma(\widetilde{B}), \qquad 0 < \beta < 1, \tag{22}$$

subject to:

$$\mathbf{1}'\widetilde{B} = 1, \tag{23}$$

with and without:

$$\widetilde{B} \geq 0. \tag{24}$$

Partition

$$\gamma(\widetilde{B}) = \begin{pmatrix} \widetilde{B} \\ \nu \end{pmatrix}, \qquad \text{with } \nu \text{ known constants;}$$

and write

$$\Gamma(\beta)\Omega\Gamma(\beta)' = V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

conformably with $\gamma(\widetilde{B})$.

Let $\#B$ denote the dimension of $\widetilde{B}$: it is the number of different gilts available. The optimal gilts structure is then characterised in the following:

THEOREM: *The problem* (22)–(23) *is solved by:*

$$\widetilde{B}_{(1)} = \widetilde{B}_{(0)} + \left(\mathbf{1}'V_{11}^{-1}\mathbf{1}\right)^{-1} \left(V_{11}^{-1}\mathbf{1}\right) \left(1 - \mathbf{1}'\widetilde{B}_{(0)}\right), \tag{25}$$

where

$$\widetilde{B}_{(0)} = -V_{11}^{-1}V_{12}\nu \tag{26}$$

is the unconstrained minimiser to (22) *alone. The problem* (22)–(24) *is solved by* $\widetilde{B}_{(2)}$ *such that for some non-negative* λ *having dimension* $\#B$*, we have:*

$$\lambda'\widetilde{B}_{(2)} = 0; \tag{27}$$

and

$$\begin{pmatrix} I & H \end{pmatrix} \begin{pmatrix} \widetilde{B}_{(2)} \\ \lambda \end{pmatrix} = \widetilde{B}_{(1)} \tag{28}$$

with

$$H \stackrel{\text{def}}{=} \left(\left[\mathbf{1}'V_{11}^{-1}\mathbf{1}\right]^{-1} V_{11}^{-1}\mathbf{1}\mathbf{1}' - I\right) V_{11}^{-1}.$$

∎

(The proof of this is given in the Technical Appendix.)

The Theorem places our earlier intuition on firmer footing. In words, it says that our uncertainty-minimizing debt structure can be found in steps. First, calculate the unconstrained minimiser $\widetilde{B}_{(0)}$ from (26). Then, find $\widetilde{B}_{(1)}$ by perturbing $\widetilde{B}_{(0)}$ using the adjustment factor that is the last term on the right-hand side of (25). This adjustment factor vanishes if we already have $\mathbf{1}'\widetilde{B}_{(0)} = 1$; direct calculation verifies that $\mathbf{1}'\widetilde{B}_{(1)} = 1$ always.

Equations (27) and (28) allow us to solve for the non-negative uncertainty-minimizing debt structure, again just by modifying the earlier $\widetilde{B}_{(1)}$. Vector λ will have an interpretation as a Lagranger multiplier. Since both λ and $\widetilde{B}$ are non-negative, the (apparently scalar) inner product condition (27) actually provides $\#B$ complementary slackness equations. When $\lambda = 0$, equation (28) collapses to just $\widetilde{B}_{(2)} = \widetilde{B}_{(1)}$; otherwise, $\widetilde{B}_{(2)}$ is a perturbation on $\widetilde{B}_{(1)}$, depending on the magnitude of λ. Note, finally, that since $\mathbf{1}'H = 0$, it follows that

$$\mathbf{1}' \begin{pmatrix} I & H \end{pmatrix} \begin{pmatrix} \widetilde{B}_{(2)} \\ \lambda \end{pmatrix} = \mathbf{1}'\widetilde{B}_{(1)}$$
$$\Longrightarrow \mathbf{1}'\widetilde{B}_{(2)} = \mathbf{1}'\widetilde{B}_{(1)} = 1.$$

Equations (27) and (28) might appear to require a numerical approximation routine to solve. However, it is a simple matter to just enumerate all $2^{\#B}$ complementary slackness possibilities, for each of which an explicit closed-form solution can be found. The Technical Appendix details this.

5. Data and empirical preliminaries

In the notation developed above, we will consider W to include, in addition to $\left(\widetilde{R}_t', G_t, T_t\right)'$, also Y output and P the rate of inflation. Thus, the dataset comprises eleven variables: six holding period returns series corresponding to different categories of government debt, the level of government spending (net of debt servicing costs), debt servicing costs, the level of tax revenue, GDP, and inflation. All the variables are quarterly and with the obvious exception of inflation are expressed in real terms.

The six categories of debt are distinguished by type (nominal and index-linked) and maturity (short, medium, and long).[9] The analysis of the type of debt is enhanced considerably by the fact that the UK government has issued index-linked debt since 1982.[10] This means that it is possible to include return data on index bonds of different maturities. This represents a significant advance on the existing literature which has tended to concentrated on the structure of US government debt, and has thus been forced to compare the real returns on nominal (and foreign currency) debt against an implicit benchmark of the certain real return offered by one-period index bonds (Bohn 1990a, 1990b).[11]

The division of debt between the three maturity bands is necessarily arbitary. For consistency with UK debt management, we followed the convention used by the Bank of England and HM Treasury (see, for example, HM Treasury 1996): short (0-7 years), medium (7-15 years) and long (15 years plus).[12] Figure 1 shows how the composition of UK government debt between nominal and index-linked debt has evolved since 1986. Although UK government debt is still heavily biased towards nominal debt, the proportion of index-linked debt has grown steadily over this period. Figures 2 and 3 show the maturity composition of these two types of debt. Note how skewed index-linked debt is towards longer-term maturities. The

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[9] Foreign currency debt was excluded from the analysis since in practice the level of foreign currency debt in the UK is determined in large part by the level of foreign exchange reserves, rather than as part of the more general financing requirements. Foreign currency debt currently accounts for less than 6% of the total market holdings of UK national debt (Bank of England, 1995a).

[10] See Bank of England (1995b) for a discussion of the history of index-linked debt in the UK.

[11] Missale (1996) considered the optimal debt structure for the UK from the perspective of tax smoothing, but did not use the data on index bonds.

[12] The sensitivity of the results to the choice of maturity bands was investigated by considering an alternative definition of 0–5, 5–10, and 10 years and above. The correlation between returns in the corresponding maturity bands was high in every case.

maturity composition of nominal debt is biased in the opposite direction, with a disproportionate quantity of nominal debt held in short term maturities. Despite this, the average maturity of UK nominal debt is significantly greater than, for example, that in the US or Germany.[13]

The holding period return (*HPR*) series for each of the six categories of government debt were derived from *HPR*s calculated for individual government bonds.[14] The *HPR* is defined as the first difference of the log of the "dirty" price of the bond, where the "dirty" price includes the interest that has accrued on the bond since the previous dividend payment.[15] The six *HPR* series were calculated by grouping the individual government bonds outstanding in each period according to their type and maturity and calculating the average *HPR* for the group of bonds falling in each debt category. The corresponding real *HPR* series were then derived by subtracting the quarterly inflation rate, as measured by the GDP deflator.

Figures 4 and 5 show the real *HPR* series for the different maturities of nominal and index-linked debt over the period 1983Q1-1996Q1. The sample period is restricted by the fact that the UK government did not start issuing indexed bonds until 1982, and the first quarter in which there were indexed bonds in each maturity band was 1982Q3.[16]

Table 1 summarises the means, variances, and correlations of the *HPR* series.

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[13] Almost 30% of UK nominal debt has a maturity in excess of 10 years, compared with 18% in the US and 11% in Germany.

[14] Some conventional bonds were excluded on the grounds that they had characteristics which might generate non-representative returns. These included irredeemable bonds, floating-rate bonds, and some bonds with very small quantities outstanding. All index-linked bonds were included.

[15] This calculation is adjusted to take account of the "ex-dividend" status bonds acquired in the period immediatedly preceeding a coupon payment. Appendix 1 provides details.

[16] *HPR*s for nominal bonds were calculated back to 1970 and this longer data series is used in some of the results reported in Section 5.

Two points are worth noting. First, as might be expected, returns on the two types of debt have variances that increase with horizon to maturity. This is consistent with economic shocks having a persistent effect on discount rates and implies that the maturity structure is an important determinant of the overall exposure of government debt to economic shocks. Second, the variance of returns on nominal debt far exceed those on index-linked debt of comparable maturities. The relative size of these variances means that even if the conditional returns on nominal debt negatively covary with shocks to the primary deficit, it need not be optimal for the debt manager to issue nominal debt. The debt manager will only do so if the impact of this negative covariance on the total (conditional) variance of the budget more than offsets the greater variance introduced by holding nominal debt.

The fiscal variables are defined in relation to central government operations: this accounts for the vast majority of public sector economic activity.[17] Tax revenues are measured by the sum of current receipts and taxes on capital. As previously noted, this measure of tax receipts includes seignorage from monetary issue. Government expenditure is defined as the sum of current and capital expenditure. This sum is adjusted in two ways in order to generate a variable conforming to the concept of government expenditure used in (11). First, government expenditure is calculated net of debt-servicing costs. Second, current expenditure includes cash expenditures on company securities, which incorporate as a negative entry privatisation proceeds. These expenditures were included so that the deficit measure corresponds most closely to the government's financing requirements.[18]

[17] The Data Appendix gives details of the definitions of the fiscal and conditioning variables.

[18] The final two variables (GDP growth and inflation) enter as auxillary conditioning variables in the VAR used to generate the conditional variance-covariance matrix. GDP is evaluated at market prices and inflation is defined by the GDP price deflator.

6. Empirical results

Table 2 shows sample statistics on public debt-weighted *HPR*s $\widetilde{R}_t = b_{t-1}R_t$: it is directly comparable to Table 1.

Because total debt b is only a slowly-moving series, the correlations in Table 2 closely match those in Table 1. The mean series are directly interpretable in terms of potential average outpayments, measured in 10^7 1990 pounds sterling. In the same units, quarterly GDP in 1990 was 1378: the averages on conventional gilts thus exceed 10% of GDP, while those on index-linked gilts are about 5%.

Once again, unconditional volatilities rise with maturity across both conventional and index-linked gilts. Although short index-linked gilts show slightly greater variability than the comparable conventionals, in general, it is index-linked gilts that have less volatile returns.

Table 3 displays details on the optimal debt portfolio for a single specification of our model. Here, we use a 4th-order VAR with only the minimal variables for the computation, i.e., the six holding-period returns (multiplied by total debt), government spending, and tax revenues. The first three lines provide intermediate calculations leading up to $\widetilde{B}_{(2)}$; the second line displays $\widetilde{B}_{(1)}$, the optimal portfolio not restricting to only non-negative debt issue. The gilts are ordered first as conventionals and then index-linked, and within each grouping, as shorts, mediums, and longs. In this example, we use a quarterly discount factor of $\beta = 0.99$, implying an annual discount rate of approximately 4%.

The optimal debt portfolio here is concentrated entirely on index-linked gilts, with close to half of the outstanding portfolio on long debt. To understand how this comes about, study the remaining lines in this Table, giving the estimated $-V_{12}\nu$, V_{11}, and V_{11}^{-1}. The first of these, $-V_{12}\nu$, shows essentially the negative of the conditional long-run covariance of holding-period returns with the primary deficit $G - T$. Roughly put, a negative entry here implies a positive (conditional) covariance, and thereby increased deficit (conditional) variability. We see negative entries on all the conventional gilts, and positive entries on all index-linked. Thus, provided individual conditional long-run variances are not too out of line relative to one another, it is unsurprising that conventionals are not in the optimal nonnegative debt portfolio.

Turn next to V_{11}. This array displays the conditional long-run variance-covariance matrix of holding-period returns. On the diagonal, the variance entries show that the index-linked instruments dominate the conventionals. Note, moreover, that conditional long-run variances for the index-linked gilts are *declining* in maturity, not increasing. This is the opposite of the pattern one finds for the unconditional volatilities, and highlights the importance of accounting for conditioning and dynamics. Holding-period returns on long index-linked gilts, multiplied by total debt outstanding, are more predictable than those on other index-linked and conventional gilts. From these observations, we see that it is no surprise why the estimated optimal debt portfolio is what it is.

Next, turn to Table 4. This shows the results for a range of empirical specifications.[19] The first line of this table simply repeats the finding from Table 3. The remaining lines vary the discount factor and use auxiliary conditioning information. The variation in optimal portfolios give some idea on the statistical imprecision of our results, given the data. Optimal portfolios do change across specifications. The third bloc, where inflation P is the auxiliary conditioning variable, shows 100% concentration of index-linked gilts at either short or medium horizons. However, studying the values of the lagrange multipliers here (not displayed) shows that in each instance the omission of the other index-linked instruments is marginal—a slight perturbation would have brought other index-linked gilts into the optimal portfolio.

Summarizing, Table 4 warns us not to put too much weight on the specifics of the optimal portfolio previously discussed (surrounding Table 3). It does confirm, however, that index-linked gilts should be favoured, and conventionals eschewed.

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[19] The reader will notice that these are all 4-th order VARs. We experimented with changing lag lengths: decreasing it by 1 did not affect the results; increasing it, we quickly ran into numerical instability due to the fairly short data expanse.

7. Conclusions

This paper has made a modest proposal for setting the structure of public debt across maturities and indexation characteristics. It departs from previous analyses in proposing that debt be used to minimise the uncertainty surrounding government policy, not—as in other work—to support tax smoothing, which we view as just one particular tax policy among many possibilities.

In this work, we have constructed a rich dataset of holding period returns and market values outstanding on UK public debt across varying maturities, for conventional and index-linked gilts.

Conditional on the historical paths for UK macroeconomic variables, we conclude that index-linked gilts make up the optimal public debt portfolio for deficit smoothing. Details vary across specifications, but in no instance did conventional gilts appear in any optimal portfolio we derived from the data.

8. Appendix

This Appendix verifies claims made in Section 2 of the paper. Begin by noticing that for any function f, for all $s \geq t+2$, and all $r \in [t+1, s]$, we have that

$$\sum_{\omega_{t+1}^{t+r}|\omega_0^t} \left[\sum_{\omega_{r+1}^s|\omega_0^r} f(\omega_0^s) \right] = \sum_{\omega_{t+1}^s|\omega_0^t} f(\omega_0^s) \tag{29}$$

(this is simply a form of the reasoning underlying the Law of Iterated Expectations).

PROPOSITION A.1: *The infinite set of equations* (3) *for all* $t \geq 0$ *and all possible histories* ω_0^t *(consistent with* ω_0*) is equivalent to the single constraint at time 0:*

$$p_0 \left[c_0 - (1-\tau_0)W(x_0)\right] + \sum_{s=1}^{\infty} \sum_{\omega_1^s|\omega_0} p_s(\omega_0^s) \left[c_s(\omega_0^s) - (1-\tau_s(\omega_0^s)W(x_s(\omega_0^s), \omega_0^s)\right]$$

$$= p_0 b_{-1,0} + \sum_{s=1}^{\infty} \sum_{\omega_1^s|\omega_0} p_s(\omega_0^s) b_{-1,s}(\omega_0^s) \tag{30}$$

PROOF: *To ease notation, define*

$$\zeta_t(\omega_0^t) \stackrel{\text{def}}{=} c_t(\omega_0^t) - (1 - \tau_t(\omega_0^t))W(x_t(\omega_0^t), \omega_0^t).$$

Assuming equation (30)*, note that in period t after ω_0^t has realized, the consumer faces the constraint:*

$$p_t(\omega_0^t)\zeta_t(\omega_0^t) + \sum_{s=t+1}^{\infty} \sum_{\omega_{t+1}^s | \omega_0^t} p_s(\omega_0^s)\zeta_s(\omega_0^s)$$
$$= p_t b_{t-1,t}(\omega_0^t) + \sum_{s=t+1}^{\infty} \sum_{\omega_{t+1}^s | \omega_0^t} p_s(\omega_0^s) b_{t-1,s}(\omega_0^s)$$

while the period after that, when ω_0^{t+1} has realized,

$$p_{t+1}(\omega_0^{t+1})\zeta_{t+1}(\omega_0^{t+1}) + \sum_{s=t+2}^{\infty} \sum_{\omega_{t+2}^s | \omega_0^{t+1}} p_s(\omega_0^s)\zeta_s(\omega_0^s)$$
$$= p_{t+1} b_{t,t+1}(\omega_0^{t+1}) + \sum_{s=t+2}^{\infty} \sum_{\omega_{t+1}^s | \omega_0^{t+1}} p_s(\omega_0^s) b_{t,s}(\omega_0^s).$$

Summing over all $\omega_{t+1}^{t+1} \mid \omega_0^t$ and using (29)*, this last equation becomes just*

$$\sum_{s=t+1}^{\infty} \sum_{\omega_{t+1}^s | \omega_0^t} p_s(\omega_0^s)\zeta_s(\omega_0^s) = \sum_{s=t+1}^{\infty} \sum_{\omega_{t+1}^s | \omega_0^t} p_s(\omega_0^s) b_{t,s}(\omega_0^s).$$

Subtracting this from the time t equation above gives

$$p_t(\omega_0^t)\zeta_t(\omega_0^t) = p_t(\omega_0^t) b_{t-1,t}(\omega_0^t) + \sum_{s=t+1}^{\infty} \sum_{\omega_{t+1}^s | \omega_0^t} p_s(\omega_0^s)(b_{t-1,s}(\omega_0^s) - b_{t,s}(\omega_0^s)),$$

i.e., equation (3) *. To show the implication the other way, write* (3) *as*

$$p_t(\omega_0^t)[\zeta_t(\omega_0^t) - b_{t-1,t}(\omega_0^t)] = \sum_{s=t+1}^{\infty} \sum_{\omega_{t+1}^s | \omega_0^t} p_s(\omega_0^s)(b_{t-1,s}(\omega_0^s) - b_{t,s}(\omega_0^s)),$$

and insert b_{-1} *on the left side to give*

$$p_t(\omega_0^t)[\zeta_t(\omega_0^t) - b_{-1,t}(\omega_0^t)] = p_t(\omega_0^t)[b_{t-1,t}(\omega_0^t) - b_{-1,t}(\omega_0^t)] + \sum_{s=t+1}^{\infty} \sum_{\omega_{t+1}^s | \omega_0^t} p_s(\omega_0^s)(b_{t-1,s}(\omega_0^s) - b_{t,s}(\omega_0^s)).$$

Summing across $\omega_1^t \mid \omega_0$ *and then over* $t \geq 0$ *gives*

$$\begin{aligned} p_0[\zeta_0 - b_{-1,0}] + \sum_{t=1}^{\infty} \sum_{\omega_1^t | \omega_0} p_t(\omega_0^t)[\zeta_t(\omega_0^t) - b_{-1,t}(\omega_0^t)] &= \sum_{t=1}^{\infty} \sum_{\omega_1^t | \omega_0} p_t(\omega_0^t)[b_{t-1,t}(\omega_0^t) - b_{-1,t}(\omega_0^t)] \\ &+ \sum_{t=0} \sum_{s=t+1}^{\infty} \sum_{\omega_1^s | \omega} p_s \omega_0^s (b_{t-1,s}(\omega_0^s) - b_{t,s}(\omega_0^s)). \end{aligned}$$

This reduces to (30) *if the left hand side is 0. Interchanging orders of summation in* t *and* s *on the last summand gives*

$$\sum_{s=1}^{\infty} \sum_{\omega_1^s | \omega} \sum_{t=0}^{s-1} p_s \omega_0^s (b_{t-1,s}(\omega_0^s) - b_{t,s}(\omega_0^s)) = \sum_{s=1}^{\infty} \sum_{\omega_1^s | \omega} p_s \omega_0^s (b_{-1,s}(\omega_0^s) - b_{s-,s}(\omega_0^s)),$$

as the sums in t *telescope out. But this last expression is simply the negative of the other term on the right side of the previous equation, thus completing the proof.* *Q.E.D.*

One of the key differences between constraint (30) in the Proposition and equations (3) is that only the portfolio holding b_{-1} at the beginning of period 0 enters the former whereas all future portfolio holdings b_t enter the latter. Thus, one can view the former as a folding back to the initial period of all the equations (3). In that initial period, of course, long instruments $b_{-1,t}$ are still held.

With this in mind, the expression given in the Proposition has a simple interpretation. The left side is the total planned expenditure (purchasing consumption good and leisure) over the entire future and in all possible histories; the right side is the total value of the initial portfolio of contingency claims. Note, again, that this equation holds not in expectation but exactly, for every state ω_0.

It only remains to show that for the government, the infinite set of equations (6) for all t and all ω_0^t is equivalent to the single constraint (7). The argument to do this is exactly the same as that in the proof of the Proposition; simply now define

$$\zeta_t(\omega_0^t) \stackrel{\text{def}}{=} \tau_t(\omega_0^t) W_t(x_t(\omega_0^t), \omega_0^t) - g_t(\omega_0^t).$$

9. Technical Appendix

The discussion surrounding (19) claimed that the linear innovations representation would be available even with non-stationarity in the form of cointegrated and difference-stationary data. Here, we derive this property explicitly. As stated in the text the result is not conceptually deep or subtle. The manipulations are convenient to record, however, as the empirical implementation uses them.

Partition the vector W as

$$W = \begin{pmatrix} X \\ Z \end{pmatrix},$$

where X is difference stationary (i.e., integrated of order 1, or $I(1)$), and Z is covariance stationary (or $I(0)$). Such a partitioning might depart from the convention used above where W_t begins with $(G_t, T_t, \widetilde{R}_t')'$, but the re-ordering to get either version of W is both obvious and trivial, and this choice saves on additional notation.[20] The nonstationary vector X might well be cointegrated; the manipulations below explicitly permit that.

Define the vector

$$Y = \begin{pmatrix} \Delta X \\ Z \end{pmatrix};$$

by construction, Y is entirely stationary. But, from whatever cointegration might exist in X, the vector Y could have rank-deficient spectral density. Vector W, by contrast, is a mixture of integrated, cointegrated, and stationary series, and thus is also ill-behaved. However, while the deficit d in (20) can be written as an inner product in W, the same is not true of Y.

If X is not cointegrated, then by the standard invertibility condition, Y has the VAR representation

$$Y_t = \sum_{j \geq 1} a(j) Y_{t-j} + \epsilon_t, \qquad E(\epsilon_t \mid Y_{t-1}, Y_{t-2}, \ldots) = 0, \tag{31}$$

and is well-approximated by an ordinary finite-order regression model, estimable from data.

When X is cointegrated, however, the sequence a in (31) does not converge, and thus (31) is not usefully approximated by any finite estimated model. But, as with standard cointegration models, the following (mixed VAR/error-correction) representation

$$Y_t = -\Phi X_{t-1} + \sum_{j \geq 1} a(j) Y_{t-j} + \epsilon_t, \qquad E(\epsilon_t \mid X_{t-1}, Y_{t-1}, Y_{t-2}, \ldots) = 0 \tag{32}$$

is well-behaved, with Φ having rank equal to X's cointegrating rank. This differs from a standard VAR in the presence of the leading term $-\Phi X(t-1)$. It differs

[20] Sometimes, one might also wish to take into account series of integration order exceeding 1. As we will see, however, those are easily incorporated in the study. We ignore them here for notational ease.

from a standard error-correction mechanism in that that leading term is not the accumulation of the left-hand side vector Y: while the accumulation of ΔX is X, the accumulation of Z has no meaning. If X is not cointegrated, then its cointegrating rank is 0, whereupon Φ vanishes and (32) collapses to (31). Equation (32) is easily estimated, by either least squares or a range of other methods.

The problem with (32) is that it does not immediately display the linear innovations representation (19), and thus does not allow us to calculate the objective function (21) analytically.[21] To that end, begin by partitioning (32) conformably into

$$\begin{pmatrix} \Delta X_t \\ Z_t \end{pmatrix} = -\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} X_{t-1} + \sum_{j\geq 1} \begin{pmatrix} a_{11}(j) & a_{12}(j) \\ a_{21}(j) & a_{22}(j) \end{pmatrix} \begin{pmatrix} \Delta X_{t-j} \\ Z_{t-j} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}. \tag{33}$$

The innovations representation for W can then be obtained from the following:

PROPOSITION: *The mixed representation* (33) *implies a unique VAR representation for the levels vector* W*:*

$$\begin{pmatrix} X_t \\ Z_t \end{pmatrix} = \sum_{j\geq 1} \begin{pmatrix} b_{11}(j) & b_{12}(j) \\ b_{21}(j) & b_{22}(j) \end{pmatrix} \begin{pmatrix} X_{t-j} \\ Z_{t-j} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix} \tag{34}$$

where

$$b_{11}(j) = \begin{cases} a_{11}(1) + I - \Phi_1, & j = 1; \\ a_{11}(j) - a_{11}(j-1), & j \geq 2 \end{cases} \tag{35}$$

$$b_{12}(j) = a_{12}(j) \tag{36}$$

$$b_{21}(j) = \begin{cases} a_{21}(1) - \Phi_2, & j = 1; \\ a_{21}(j) - a_{21}(j-1) & j \geq 2 \end{cases} \tag{37}$$

$$b_{22}(j) = a_{22}(j), \tag{38}$$

[21] One can, of course, simulate (32) on a machine, but the statement just made remains true.

with the disturbances in (33) and (34) identical.

PROOF OF PROPOSITION: *From (33) write out the ΔX equation explicitly:*

$$X_t - X_{t-1} = -\Phi_1 X_{t-1} + \sum_{j\geq 1} a_{11}(j)\left(X_{t-j} - X_{t-j-1}\right) + \sum_{j\geq 1} a_{12}(j) Z_{t-j} + \epsilon_{1,t}.$$

Rearrange to give:

$$\begin{aligned} X_t &= \left(a_{11}(1) + I - \Phi_1\right) X_{t-1} + \sum_{j\geq 2}\left(a_{11}(j) - a_{11}(j-1)\right) X_{t-j} \\ &\quad + \sum_{j\geq 1} a_{12}(j) Z_{t-j} + \epsilon_{1,t} \\ &= \sum_{j\geq 1} b_{11}(j) X_{t-j} + \sum_{j\geq 1} b_{12}(j) Z_{t-j} + \epsilon_{1,t}, \end{aligned}$$

defining b_{11} and b_{12} as in (35) and (36). Next, again from (33), write out the Z equation explicitly:

$$\begin{aligned} Z_t &= -\Phi_2 X_{t-1} + \sum_{j\geq 1} a_{21}(j)\left(X_{t-j} - X_{t-j-1}\right) \\ &\quad + \sum_{j\geq 1} a_{22}(j) Z_{t-j} + \epsilon_{2,t} \\ &= \left(a_{21}(1) - \Phi_2\right) X_{t-1} + \sum_{j\geq 2}\left(a_{21}(j) - a_{21}(j-1)\right) X_{t-j} \\ &\quad + \sum_{j\geq 1} a_{22}(j) Z_{t-j} + \epsilon_{2,t} \\ &= \sum_{j\geq 1} b_{21}(j) X_{t-j} + \sum_{j\geq 1} b_{22}(j) Z_{t-j} + \epsilon_{2,t}, \end{aligned}$$

with b_{21} and b_{22} as in (37) *and* (38). *Q.E.D.*

The VAR representation given in (34) mixes both $I(1)$ and $I(0)$ variables (in levels), and with the coefficients non-trivial. In our work, this representation is estimated, not in the form (34), but in the earlier (33). It is then reorganized by the Proposition into (34). The resulting levels representation can then be manipulated in the usual way, ignoring the differing orders of integration and cointegration as the restrictions implied by those are automatically built into the structure of coefficients b.

From this VAR representation, rewritten explicitly as

$$W_t = \sum_{j\geq 1} b(j)W_{t-j} + \epsilon_t, \qquad E(\epsilon_t \mid W_{t-1}, W_{t-2}, \dots) = 0,$$

we compute the linear innovations representation:

$$\forall j \geq 1: \qquad W_{t+j} = \sum_{k=0}^{j-1} c(k)\epsilon_{t+j-k} + \varphi(\mathit{info}(t))$$
$$\Longrightarrow W_{t+j} \mid \mathit{info}(t) = \sum_{k=0}^{j-1} c(k)\epsilon_{t+j-k},$$

since $\mathit{info}(t) = \{W_t, W_{t-1}, \dots\}$, the function φ is known, and the sequence c depends only on sequence b.

Turn now to the Theorem in the text.

PROOF OF THEOREM: *For the unconstrained minimiser $\widetilde{B}_{(0)} = -V_{11}^{-1}V_{12}\nu$, note that for arbitrary $\widetilde{B}$, the difference*

$$\begin{aligned}
\begin{pmatrix} \widetilde{B}' & \nu' \end{pmatrix} V \begin{pmatrix} \widetilde{B} \\ \nu \end{pmatrix} - \begin{pmatrix} \widetilde{B}'_{(0)} & \nu' \end{pmatrix} V \begin{pmatrix} \widetilde{B}_{(0)} \\ \nu \end{pmatrix}& \\
&= \widetilde{B}'V_{11}\widetilde{B} + \widetilde{B}'V_{12}\nu + \nu'V_{21}\widetilde{B} + \nu'V_{21}V_{11}^{-1}V_{12}\nu \\
&= \begin{pmatrix} \widetilde{B}' & \nu' \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{21}V_{11}^{-1}V_{12} \end{pmatrix} \begin{pmatrix} \widetilde{B} \\ \nu \end{pmatrix} \\
&\geq 0
\end{aligned}$$

since

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{21}V_{11}^{-1}V_{12} \end{pmatrix} = \begin{pmatrix} V_{11}^{1/2} \\ V_{21}V_{11}^{-1/2} \end{pmatrix} \begin{pmatrix} V_{11}^{1/2} & V_{11}^{-1/2}V_{12} \end{pmatrix}$$

is clearly positive semi-definite (with $V_{11}^{1/2}$ *the symmetric square root of* V_{11}*). Next, for* $\widetilde{B}_{(1)}$*, form the Lagrangean*

$$\mathcal{L}_1 = -\begin{pmatrix} \widetilde{B}' & \nu' \end{pmatrix} V \begin{pmatrix} \widetilde{B} \\ \nu \end{pmatrix} - 2\mu[1 - \mathbf{1}'\widetilde{B}].$$

Differentiating with respect to $\widetilde{B}$ *gives*

$$\frac{1}{2}\frac{\partial \mathcal{L}_1}{\partial \widetilde{B}} = -V_{11}\widetilde{B} - V_{12}\nu + \mu\mathbf{1} = 0 \tag{39}$$

and with respect to μ *gives*

$$-\frac{1}{2}\frac{\partial \mathcal{L}_1}{\partial \mu} = 1 - \mathbf{1}'\widetilde{B} = 0. \tag{40}$$

Rewrite (39) *as*

$$\mathbf{1}\mu = V_{11}\widetilde{B} + V_{12}\nu$$

so that from (40),

$$\begin{aligned} \mathbf{1}'V_{11}^{-1}\mathbf{1}\nu &= \mathbf{1}'\widetilde{B} + \mathbf{1}'V_{11}^{-1}V_{12}\nu \\ &= 1 - \mathbf{1}'\widetilde{B}_{(0)} \\ \implies \mu &= \left(\mathbf{1}'V_{11}^{-1}\mathbf{1}\right)^{-1}\left(1 - \mathbf{1}'\widetilde{B}_{(0)}\right). \end{aligned}$$

Substituting this in (39) *gives the optimal* $\widetilde{B}$ *as*

$$\begin{aligned} \widetilde{B}_{(1)} &= -V_{11}^{-1}V_{12}\nu + V_{11}^{-1}\mathbf{1}\mu \\ &= \widetilde{B}_{(0)} + \left(\mathbf{1}'V_{11}^{-1}\mathbf{1}\right)^{-1}\left(V_{11}^{-1}\mathbf{1}\right)\left(1 - \mathbf{1}'\widetilde{B}_{(0)}\right). \end{aligned}$$

Finally, for $\widetilde{B}_{(2)}$, define the Lagrangean

$$\mathfrak{L}_2 = -\begin{pmatrix} \widetilde{B}' & \nu' \end{pmatrix} V \begin{pmatrix} \widetilde{B} \\ \nu \end{pmatrix} - 2\mu[1 - \mathbf{1}'\widetilde{B}] + 2\lambda'\widetilde{B}.$$

Differentiating with respect to $\widetilde{B}$ gives

$$\frac{1}{2}\frac{\partial \mathfrak{L}_2}{\partial \widetilde{B}} = -V_{11}\widetilde{B} - V_{12}\nu + \mu\mathbf{1} + \lambda = 0 \tag{41}$$

and with respect to μ gives

$$-\frac{1}{2}\frac{\partial \mathfrak{L}_2}{\partial \mu} = 1 - \mathbf{1}'\widetilde{B} = 0. \tag{42}$$

Rewrite (41) *as*

$$\mathbf{1}\mu = V_{11}\widetilde{B} + V_{12}\nu - \lambda$$

so that from (42),

$$\begin{aligned} \mathbf{1}'V_{11}^{-1}\mathbf{1}\nu &= \mathbf{1}'\widetilde{B} + \mathbf{1}'V_{11}^{-1}V_{12}\nu - \mathbf{1}'V_{11}^{-1}\lambda \\ &= 1 - \mathbf{1}'\widetilde{B}_{(0)} - \mathbf{1}'V_{11}^{-1}\lambda \\ \Longrightarrow\ \mu &= \left(\mathbf{1}'V_{11}^{-1}\mathbf{1}\right)^{-1}\left(1 - \mathbf{1}'\widetilde{B}_{(0)}\right) - \left(\mathbf{1}'V_{11}^{-1}\mathbf{1}\right)^{-1}\mathbf{1}'V_{11}^{-1}\lambda. \end{aligned}$$

Rearrange (41) *as*

$$\begin{aligned} V_{11}\widetilde{B} - \lambda &= -V_{12}\nu + \mathbf{1}\mu \\ \Longrightarrow\ \widetilde{B} - V_{11}^{-1}\lambda &= \widetilde{B}_{(0)} + V_{11}^{-1}\mathbf{1}\mu, \end{aligned}$$

so that substituting in the previous expression for μ yields

$$\begin{aligned} \widetilde{B} - V_{11}^{-1}\lambda = \widetilde{B}_{(0)} &+ \left(\mathbf{1}'V_{11}^{-1}\mathbf{1}\right)^{-1}V_{11}^{-1}\mathbf{1}\left(1 - \mathbf{1}'\widetilde{B}_{(0)}\right) \\ &- \left(\mathbf{1}'V_{11}^{-1}\mathbf{1}\right)^{-1}V_{11}^{-1}\mathbf{1}\mathbf{1}'V_{11}^{-1}\lambda. \end{aligned}$$

Collecting terms in λ and denoting the optimal $\widetilde{B}$ as $\widetilde{B}_{(2)}$ then gives

$$\widetilde{B}_{(2)} + H\lambda = \widetilde{B}_{(1)}$$

or

$$\begin{pmatrix} I & H \end{pmatrix} \begin{pmatrix} \widetilde{B}_{(2)} \\ \lambda \end{pmatrix} = \widetilde{B}_{(1)}.$$

Q.E.D.

Finally, turn to explicit solution of equations (27) and (28). Of the $(\#B) \times 2$ entries in $(\widetilde{B}', \lambda')'$, there are only $2^{\#B}$ possibilities whereby at least one of $\widetilde{B}_j$ or the matching λ_j is necessarily zero. Encode these possibilities by the binary representations of the integers in $\{ 0, 1, \ldots , (2^{\#B} - 1) \}$, such that, without loss of generality, a 0 in the j-th digit of the binary expansion of the integer indicates the corresponding $\widetilde{B}_j$ is 0—the matching λ_j is then unrestricted and needs to be solved for from (28).

More precisely, let $\omega \in \{ 0, 1, \ldots , 2^{\#B} - 1 \}$ have the binary expansion

$$\omega = \omega_1 \omega_2 \cdots \omega_{\#B}.$$

Call M_ω the matrix formed from $\begin{pmatrix} I & H \end{pmatrix}$ in (28) by deleting the j-th column if $\omega_j = 0$ and the $(\#B + j)$-th column if $\omega_j = 1$. Evidently, M_ω is square, $\#B$ by $\#B$. Similarly, let v_ω be the vector formed from $(\widetilde{B}', \lambda')'$ by deleting the j-th entry if $\omega_j = 0$ and the $(\#B + j)$-th column if $\omega_j = 1$. The operation is exactly the same as that performed when constructing M_ω. Then, equation (28) reduces to

$$\begin{aligned} M_\omega v_\omega &= \widetilde{B}_{(1)} \\ \Longrightarrow \; v_\omega &= M_\omega^{-1} \widetilde{B}_{(1)}. \end{aligned}$$

We then reconstitute $(\widetilde{B}', \lambda')'$ from v_ω.

If v_ω is not all non-negative, then the resulting $(\widetilde{B}', \lambda')'$ is not a solution. For our problem, given the empirically-derived estimates for $\Gamma(\beta)$ and Ω, there always exists a unique non-negative optimal gilts structure.

10. Data appendix

A total of 241 conventional bonds and 18 index-linked bonds were used to calculate the holding period returns. The smallest number of conventional bonds considered in any quarter was 50, compared with 7 for index-linked bonds. Individual maturity bands of conventional bonds always included at least 8 different bonds, compared with just 1 for index bonds. Average maturity in each of the categories over the sample period were as follows:

	Average maturity
Conventional: Short	3.22 years
Medium	10.78 years
Long	19.77 years
Index-linked: Short	3.35 years
Medium	11.50 years
Long	24.71 years

The *HPR*s for individual bonds were calculated using the formula:

$$HPR_t = \log\left(\frac{P_t + A_t}{P_{t-1}}\right)$$

where P_t is the dirty price (including accrued interest) in period t, and A_t is an adjustment factor constructed to account for ex-dividend effects.

UK bond prices are typically quoted 'cum dividend' which means that the most recent holder of the bond is entitled to the coupon payment. However, 37 calendar days before the coupon is due, a bond starts being quoted 'ex-dividend'; this means that the purchase of the bond during this period does not entitle the investor to the next coupon payment, instead the coupon is paid to the investor who held the bond at the end of the 'cum-dividend' period.[22] Therefore, in or-

[22] The ex-dividend period was reduced to 7 working days in January 1996.

der to calculate the appropriate holding period return for the period immediately preceding the payment of a coupon, an adjustment factor has to be calculated.

For example, suppose a bond pays a coupon in January and July and an investor buys the bond at the end of November (i.e., during the 'cum dividend' period) and sells it an the end of December (during the 'ex dividend' period). The total profit associated with the investor holding the bond between November and December will be given by the difference between the price obtained when the bond was sold and the price paid at the moment of purchase, *plus* the present value of the coupon paid in January. Therefore, an adjustment needs to be added to the dirty price prevailing at the end of December, according to the formula:

$$A_t = \begin{cases} 0, & \text{if no cash flow occurs between } t-1 \text{ and } t \\ \frac{C_t}{1+(Libor_t^{4\tau})/4} & \text{otherwise,} \end{cases}$$

where C_t is the cash-flow between $t-1$ and t; $Libor_t$ is the 3-month London Inter-Bank Offered Rate at t; and τ is the time in years until the cashflow is received.

Fiscal variables

The data source for the fiscal variables is CSO (Central Statistical Office), which became the Office of National Statistics in April 1996, merging with the Office of Population Censuses and Surveys. We base our definitions on the summary account of the central government, in Table 7.1 of the annual CSO publication "United Kingdom National Accounts". The only exception is cash expenditure on company securities, in Table 7.4 "Central government: Transactions in financial assets and liabilities" (in the same CSO publication). In the main text, we defined the concepts we use for government expenditure G, tax revenues T and, hence, the primary deficit $G - T$. Here, we provide the details of the construction of the fiscal variables and their CSO codes.

Net government expenditure (G) (in nominal terms)
We take (following CSO abbreviations):

$$G = ACHB + ACID - ACHL + ACKG,$$

where

$$\begin{aligned} ACHB &= \text{total current expenditure (inclusive of debt-service costs)} \\ ACID &= \text{total capital expenditure} \\ ACHL &= \text{debt-service costs} \\ ACKG &= \text{cash expenditure on company securities.} \end{aligned}$$

The last variable $ACKG$ is negative when company securities sales exceed acquisitions, i.e. during the period characterised by relatively high privatisation receipts.

Total tax revenues (T) (in nominal terms)
We take (again, following CSO abbrevations):

$$T = ACGA + GTAD$$

where

$$\begin{aligned} ACGA &= \text{total current receipts} \\ GTAD &= \text{taxes on capital and other capital receipts.} \end{aligned}$$

The fiscal variables enter the analysis in real terms, where inflation is defined at the GDP at market prices implied deflator (code $DJDT$). The reason for using the GDP deflator, rather than, for example, RPIX or RPI, is that this variable reflects changes in the price level of all activity in the whole economy. We think, therefore, that it is more appropriate than changes in consumer prices for calculating real figures on fiscal variables.

Finally, real GDP and inflation enter the VAR as auxiliary (conditioning) variables. Inflation is measured as the first difference of the GDP deflator (in logarithms). Real GDP is defined as the ratio between nominal GDP and the GDP deflator, where for nominal GDP we use the data on GDP evaluated at market prices (code *CAOB*), in Table A.2 "Gross domestic product by category of expenditure" of the CSO publication "UK Economic Accounts: A quarterly supplement to Economic Trends."

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Table 1: Real Quarterly Holding Period Returns (%), 1982Q3–1996Q1

	Conventional			Index-linked		
	Short	Medium	Long	Short	Medium	Long
Mean	1.30	1.63	1.50	0.82	0.83	0.54
Std. Dev.	2.17	4.51	5.47	1.85	3.26	4.19
	Correlations					
Conventional						
Short		0.94	0.87	0.69	0.66	0.66
Medium			0.97	0.63	0.66	0.70
Long				0.53	0.57	0.66
Index-linked						
Short					0.77	0.66
Medium						0.95

Table 2: $\widetilde{R}_t = b_{t-1}R_t(\times 10^7$1990 Pounds Sterling), 1982Q3–1994Q4

	Conventional			Index-linked		
	Short	Medium	Long	Short	Medium	Long
Mean	167.5	180.0	171.2	78.8	78.8	48.6
Std. Dev.	187.8	507.4	619.6	195.6	338.4	442.2
	Correlations					
Conventional						
Short		0.76	0.70	0.42	0.43	0.40
Medium			0.97	0.49	0.61	0.67
Long				0.40	0.54	0.64
Index-linked						
Short					0.74	0.63
Medium						0.95

Table 3: An optimal portfolio
(VAR includes constant, time trend, and quarterly seasonals)

Minimum VAR, 4 lags, $\beta = 0.99$

$$\widetilde{B}_{(0)} = (\ 0.08,\ \ 0.71,\ \ -0.68,\ \ -1.73,\ \ 2.06,\ \ -0.98\ )$$
$$\widetilde{B}_{(1)} = (\ -0.73,\ \ -1.25,\ \ 0.87,\ \ 2.79,\ \ -1.50,\ \ 0.81\ )$$
$$\widetilde{B}_{(2)} = (\ 0.00,\ \ 0.00,\ \ 0.00,\ \ 0.17,\ \ 0.34,\ \ 0.49\ )$$

$$-V_{12} \times \nu = \begin{pmatrix} -3.8 \\ -4.4 \\ -5.7 \\ 0.8 \\ 2.4 \\ 1.0 \end{pmatrix} \times 10^5$$

$$V_{11} = \begin{pmatrix} 15.0 & & & & & \\ 18.6 & 29.9 & & & & \\ 16.2 & 28.7 & 30.8 & & & \\ 7.6 & 10.8 & 7.1 & 8.0 & & \\ 2.6 & 6.7 & 4.4 & 6.1 & 7.5 & \\ -0.6 & 2.5 & 3.1 & 1.4 & 4.4 & 5.4 \end{pmatrix} \times 10^5$$

$$V_{11}^{-1} = \begin{pmatrix} 3.9 & & & & & \\ 0.0 & 6.9 & & & & \\ -0.7 & -5.1 & 4.0 & & & \\ -9.7 & -7.4 & 7.1 & 33.6 & & \\ 9.7 & 3.7 & -4.4 & -30.0 & 28.6 & \\ -4.5 & -1.5 & 1.8 & 14.1 & -13.8 & 7.0 \end{pmatrix} \times 10^{-5}$$

Table 4: Optimal portfolios
(VARs include 4 lags, constant, time trend, and quarterly seasonals)

		Conventional			**Index-linked**		
β	auxiliary	Short	Medium	Long	Short	Medium	Long
0.99					0.17	0.34	0.49
0.95					0.40	0.03	0.57
0.99	Y				0.19	0.81	
0.95	Y					0.20	0.80
0.99	P				1.00		
0.95	P					1.00	
0.99	Y, P					0.10	0.90
0.95	Y, P					0.34	0.66

Figure 1: Debt structure: nominal and index - linked split

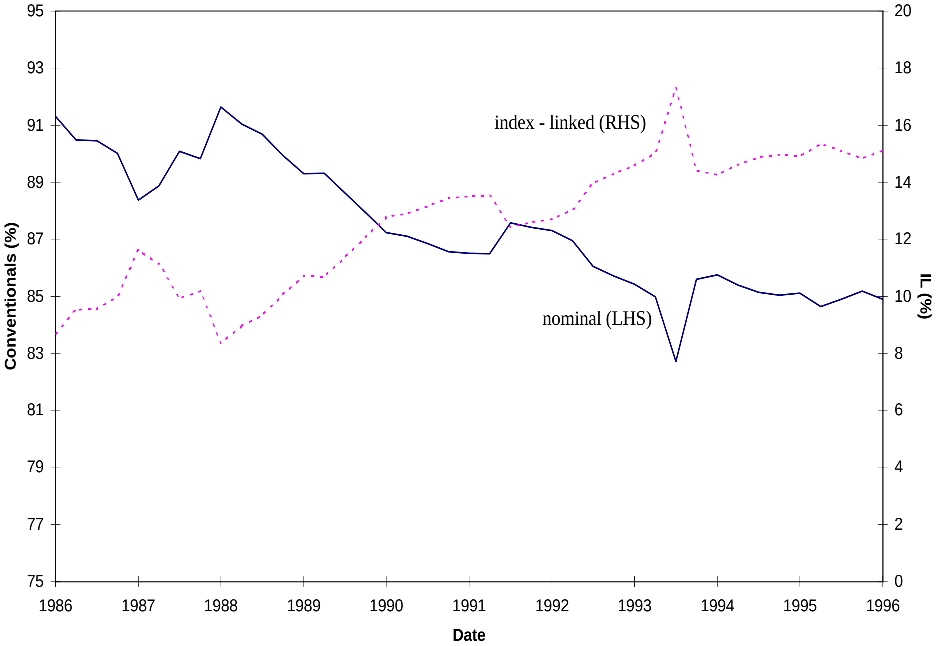

Figure 2: Maturity structure of conventional bonds

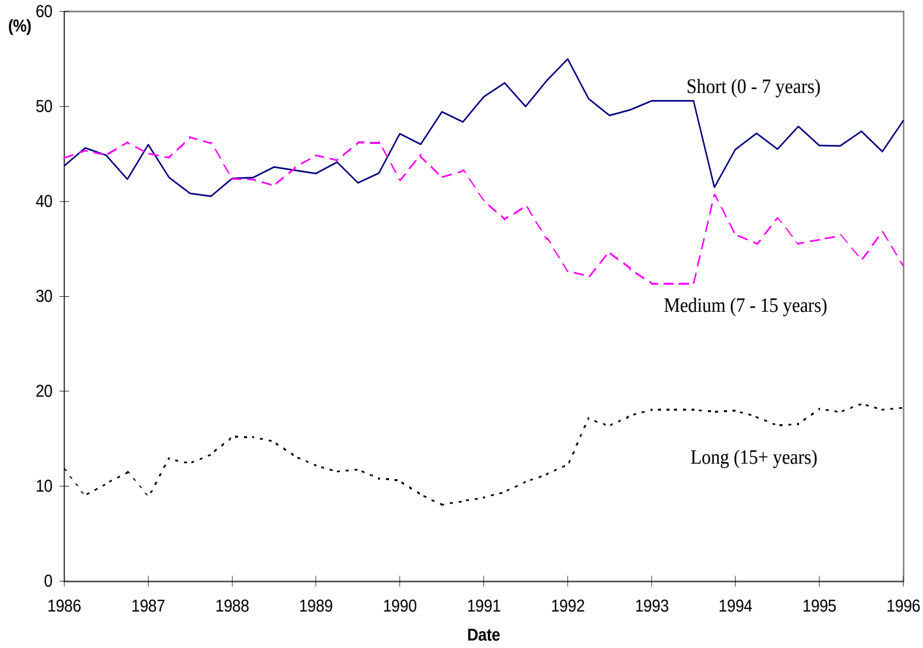

Figure 3: Maturity structure of index - linked bonds

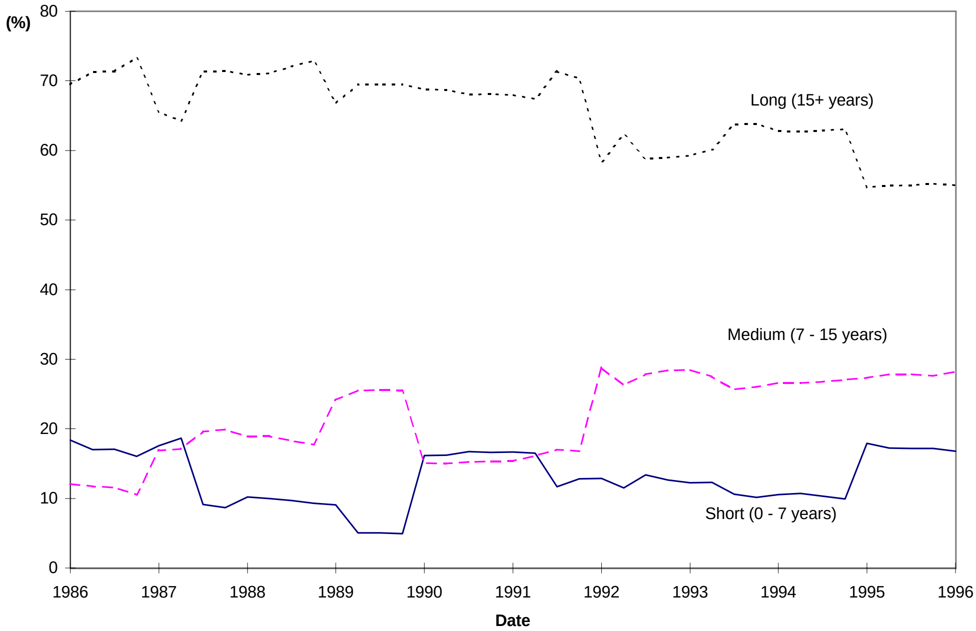

Figure 4: Holding period returns: Conventional

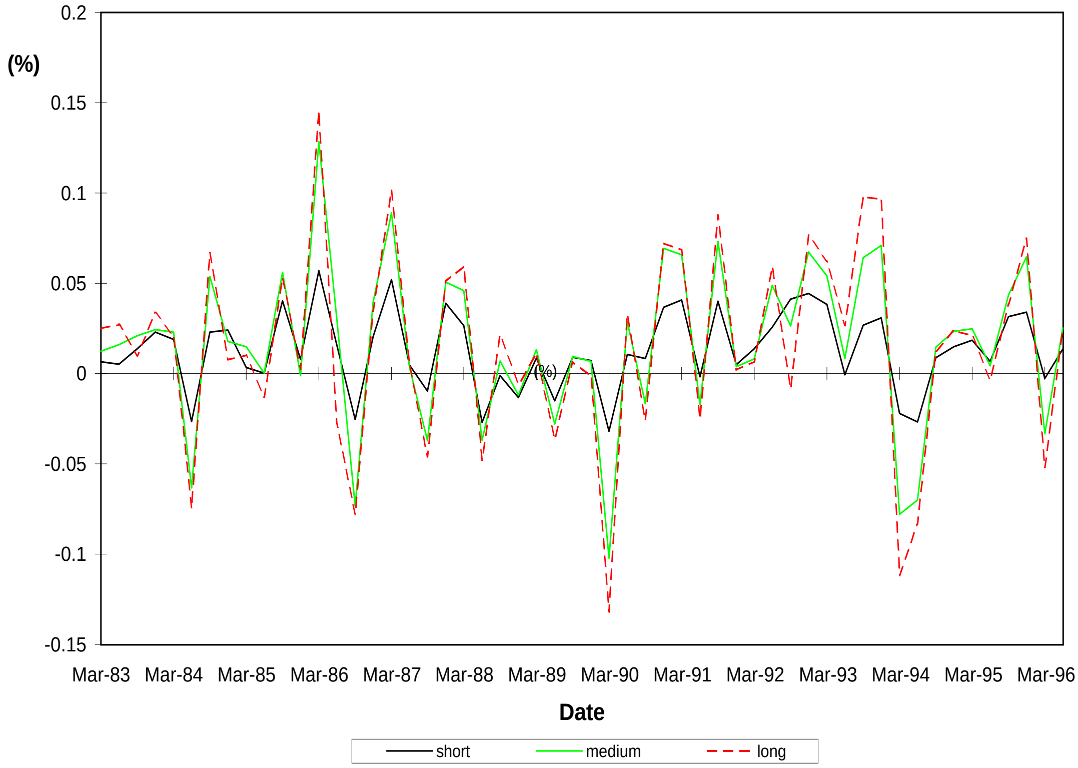

Figure 5: Holding period returns: Index-linked