

Does Competition Solve the Hold-up Problem?*

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Abstract. In an environment in which both buyers and sellers can undertake match specific investments, the presence of market competition for matches may solve hold-up and coordination problems generated by the absence of complete contingent contracts. In particular, this paper shows that when matching is assortative and sellers' investments precede market competition then investments are constrained efficient. One equilibrium is efficient with efficient matches but also there can be additional equilibria with coordination failures. Different types of inefficiency arise when sellers undertake investment before market competition. These inefficiencies lead to sellers' under-investments due to a hold-up problem but, when competition is at its peak, there is a unique equilibrium of the competition game with efficient matches — no coordination failures — and the aggregate hold-up inefficiency is small in a well defined sense independent of market size.

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1. Introduction

A central concern for economists is the extent to which competitive market systems are efficient and, in the idealized Arrow-Debreu model of general equilibrium, efficiency follows under mild conditions, notably the absence of externalities. But in recent years, economists have become interested in studying market situations less idealized than in the Arrow-Debreu set-up and in examining the pervasive inefficiencies that may exist.¹ This paper studies a market situation where there are two potential inefficiencies — these are often referred to as the “hold-up problem” and as “coordination failures”. An important part of our analysis will be to examine the connection between, as well as the extent of, the inefficiencies induced by these two problems and whether market competition may solve them.

The hold-up problem applies when a group of agents, e.g. a buyer and a seller, share some surplus from interaction and when an agent making an investment is unable to receive all the benefits that accrue from the investment. The existence of the problem is generally traced to incomplete contracts: with complete contracts, the inefficiency induced by the failure to capture benefits will not be permitted to persist. In the standard set-up of the problem, investments are chosen before agents interact and contracts can be determined only when agents meet. Prior investments will be a sunk cost and negotiation over the division of surplus resulting from an agreement is likely to lead to a sharing of the surplus enhancement made possible by one agent’s investment (Williamson 1985, Grout 1984, Grossman and Hart 1986, Hart and Moore 1988).

Coordination failures arise when a group of agents can realise a mutual gain only by a change in behaviour by each member of the group. For instance, a buyer may receive the marginal benefits from an investment when she is matched with any particular seller, so there is no hold-up problem, but she may be inefficiently matched with a seller; the incentive to change the match may not exist because mutual gains may be realised only if the buyer to be displaced is willing to alter her investment to

¹See Hart (1995) and Holmström (1999) for an extensive discussion of these inefficiencies.

make it appropriate for the new matching.

What happens if agent's interaction is through the marketplace? In an Arrow-Debreu competitive model, complete markets, with price-taking in each market, are assumed; if an agent chooses investment *ex-ante*, every different level of investment may be thought of as providing the agent with a different good to bring to the market (Makowski and Ostroy 1995). If the agent wishes to choose a particular level of investment over some other, and the "seller" he trades with also prefers to trade with the agent in question, rather than with an "identical" agent with another investment level, then total surplus to be divided must be maximized by the investment level chosen: investment will be efficiently chosen and there is no hold-up problem. In this situation, the existence of complete markets implies that agents know the price that they will receive or pay whatever the investment level chosen: complete markets imply complete contracts. In addition, as long as there are no externalities, coordination failures will not arise as the return from any match is priced in the market and this price is independent of the actions of agents not part to the match.

An unrealistic failure of the Arrow-Debreu set-up is that markets are assumed to exist for every conceivable level of investment, irrespective of whether or not trade occurs in such a market. However, if *ex-ante* investments are specific to a particular trade in most of these markets there will be no trade. It is then far-fetched to assume that agents will believe that they can trade in inactive markets and, more importantly, that a competitive price will be posted for such markets.

The purpose of this paper is to investigate the efficiency of investments when the trading pattern and terms of trade are determined explicitly by the competition of buyers and sellers. To ensure that there are no inefficiencies resulting from market power, a model of Bertrand competition is analyzed where some agents invest prior to trade; however, this does not rule out the dependence of the pattern of outcomes on the initial investment of any agent and the analysis concentrates on the case of a finite number of traders to ensure this possibility. Contracts are the result of competition in the marketplace and we are interested in the degree to which the hold-up problem and coordination problems are mitigated by contracts that result

from Bertrand competition. In this regard, it should be said that we shall not permit Bertrand competition in contingent contracts; in our analysis, contracts take the form of an agreement between a buyer and a seller to trade at a particular price. We are thus investigating the efficiency of contracts implied by a simple trading structure rather than attempting explicitly to devise contracts that help address particular problems (e.g. Aghion, Dewatripont, and Rey 1994, Maskin and Tirole 1999, Segal and Whinston 1998).

We will also restrict attention to markets where the Bertrand competitive outcome is robust to the way that markets are made to clear. To be specific, we assume that buyers and sellers can be ordered by their ability to generate surplus with a complementarity between buyers and sellers. This gives rise to assortative matching in the quality of buyers and sellers without requiring any further restriction on the market clearing process. With investment choices, the quality of buyers and/or sellers is assumed to depend on such investments. This set-up has the virtue that, as we will show, the Bertrand outcome is always efficient when investment levels are not subject to choice.

We first consider a world in which only buyers' quality depends on their ex-ante investments, sellers' qualities being exogenously given. In this case we demonstrate that buyers' investment choices are constrained efficient. In particular, for a given equilibrium match, a buyer bids just enough to win the right to trade with a seller and, if he were to have previously enhanced his quality and the value of the trade by extra investment, he would have been able to win the right with the same bid, as viewed by the seller, and so receive all the marginal benefits of the extra investment. We are able to extend this result to show that, with other agents' behaviour fixed, buyers make efficient investment choices even when they recognise that these actions will lead to a change in match. A consequence of this is that an outcome where all buyers choose efficient investments is an equilibrium in the model.

When the returns of investments in terms of buyers' quality are not too high it is possible that a buyer might undertake a high investment with the sole purpose of changing the seller with whom he will be matched and a byproduct of this will be

that another buyer is deterred from undertaking investment appropriate to this match. This may lead to inefficient equilibrium matches. In such an environment, hold-up problems are solved and the only inefficiencies left are due to buyers' pre-emption strategies when choosing their investments: inefficiencies are due to coordination failures. We show that these inefficiencies will not arise if the returns from investments differ enough across buyers.²

We then consider a world in which the sellers' quality depends on their ex-ante investments. In this case we indeed show that sellers' investments are inefficient. However, we are able to show that the extent of the inefficiency is limited.

On the one hand, when the competition among buyers for a match is most intense the overall inefficiency in a market is less than that which could result from an under-investment by one (the best) seller in the market with all other sellers making efficient investments. This result holds irrespective of the number of buyers or sellers in the market. The feature of the Bertrand competition game that determines the intensity of the competition among buyers is the sequential order in which sellers select their partner to the match. If this order is determined, at an early stage of the game, by the competition among sellers then we demonstrate that, in equilibrium, the order will be such that competition among buyers will be most intense — provided that the returns from sellers' investments differ enough across matches. In other words, competition among sellers lead to a high intensity of the competition among buyers for a match.

On the other hand, surprisingly in this case, when competition among buyers for a match is most intense all coordination problems are solved and the equilibrium matches are the efficient ones: the ordering of the sellers' qualities generated by ex-ante investments coincides with the ordering of the sellers' innate qualities. The reason for this is that sellers only reap those gains from an investment that would accrue if they were to be matched with the buyer who is the runner-up in the competitive bidding process. Critically, a seller who, through investment, changes his place

²For an analysis of how market competition may fail to solve coordination problems see also Hart (1979), Cooper and John (1988) and Makowski and Ostroy (1995).

in the quality ranking does not by that change necessarily alter the runner-up and the seller will ignore gains and losses that come purely from a change of match. Thus, it is the blunted (inefficient) incentives created by a hold-up problem that remove the inefficiencies that come from coordination failures.

The structure of the paper is as follows. After a discussion of related literature in the next section, Section 3 lays down the basic model and the extensive form of the Bertrand competition game between buyers and sellers. It is then shown in Section 4 that, with fixed investments, the competition game gives rise to an efficient outcome — buyers and sellers match efficiently. Section 5 then investigates the efficiency properties of the model where buyers undertake ex-ante investments before competition occurs. We show that buyers' investments are efficient given equilibrium matches and that the efficient outcome is always an equilibrium. However, depending on parameters, we show that equilibria with coordination failures may arise that lead to inefficient matches. We then consider in Section 6 the model in which the sellers undertake ex-ante investments. We first characterize the inefficient investment choices that will be made. We then show in Section 7 that in equilibrium sellers' competition raises the intensity of the buyers' competition for a match to its peak. When this is the case the inefficiencies generated by sellers' underinvestment are limited in a well defined sense. Section 8 provides concluding remarks.

2. Related Literature

The literature on the hold-up problem has mainly analyzed the bilateral relationship of two parties that may undertake match specific investments in isolation (Williamson 1985, Grout 1984, Grossman and Hart 1986, Hart and Moore 1988). In other words, these papers identify the inefficiencies that the absence of complete contingent contracts may induce in the absence of any competition for the parties to the match.³

³A notable exception is Bolton and Whinston (1993). This is the first paper to analyze an environment in which an upstream firm (a seller) trades with two downstream firms (two buyers) that undertake ex-ante investments. One of the cases they analyze coincides with the Bertrand competition outcome we identify in our model. However, given that this case of non-integration when only one buyer can be served arises only with an exogenously given probability and that in

This literature identifies the institutional (Grossman and Hart 1986, Hart and Moore 1990, Aghion and Tirole 1997, Rajan and Zingales 1998) or contractual (Aghion, Dewatripont, and Rey 1994, Maskin and Tirole 1999, Segal and Whinston 1998, Che and Hausch 1999) devices that might reduce and possibly eliminate these inefficiencies. We differ from this literature in that we do not alter either the institutional or contractual setting in which the hold-up problem arises but rather analyze how competition among different sides of the market may eliminate the inefficiencies associated with such a problem.⁴

The literature on bilateral matching, on the other hand, concentrates on the inefficiencies that arise because of frictions present in the matching process. These inefficiencies may lead to market power (Diamond 1971, Diamond 1982), unemployment (Mortensen and Pissarides 1994) and a class structure (Burdett and Coles 1997, Eeckhout 1999). A recent development of this literature shows how efficiency can be restored in a matching environment thanks to free entry into the market (Roberts 1996, Moen 1997) or Bertrand competition (Felli and Harris 1996). We differ from this literature in that we abstract from any friction in the matching process and focus on the presence of match specific investments by either side of the market.

A small recent literature considers investments in a matching environment. Some of the papers focus on general investments that may be transferred across matches and identify the structure of contracts (MacLeod and Malcomson 1993) or the structure of competition (Holmström 1999) and the market structure (Acemoglu and Shimer 1999, Spulber 2000) that may lead to efficiency. Other papers (Ramey and Watson 1996, Acemoglu 1997) focus on the inefficiencies induced on parties' investments by the presence of an exogenous probability that the match will dissolve. These inefficiencies arise in the presence of incomplete contracts (Ramey and Watson 1996) or even in

case both buyers can be served the gains from trade are equally shared among the seller and the two buyers in equilibrium both buyers under-invest. In other words, the way the surplus is shared in the absence of shortage and the focus on the competition among only two buyers greatly limits the efficiency enhancing effect of competition that is the main focus of our analysis.

⁴It should be said that Che and Hausch (1996) suggests the possibility that competition may enhance parties' incentives to undertake specific investments when involved in a hold-up problem.

the presence of complete but bilateral contracts (Acemoglu 1997).⁵ A recent paper by Kranton and Minehart considers, instead, the efficiency of investments in the competitive structure itself (Kranton and Minehart 2000); specifically, markets are limited by the networks that agents create through investment. Finally, two recent papers (Burdett and Coles 1999, Peters and Siow 2000) focus on the efficiency of ex-ante investments in a model in which utility is not transferable across the parties to a match, in other words they analyze marriage problems.

The two papers closest to our analysis are Cole, Mailath, and Postlewaite (2001a) and Cole, Mailath, and Postlewaite (2001b). These are the first papers to provide a detailed analysis of specific investments and market competition for matches. In particular, both papers assume that the two sides of a market first undertake match specific investments and then compete in the market place for a match. The investment choice is modelled as a non-cooperative decision while the matching process is modelled as a cooperative assignment game. Both papers focus on the core of this assignment game. The two sides of the market are assumed to be heterogeneous. In Cole, Mailath, and Postlewaite (2001b) there is a continuum of different types of individuals on both sides of the market. As a result competition for matches occurs among individuals that, before undertaking the investment, are almost perfect substitutes. Conversely, in Cole, Mailath, and Postlewaite (2001a) there is a finite number of different types of individuals on both sides of the market. Hence competition occurs among individuals that in terms of their innate characteristics are, potentially, imperfect substitutes.

Therefore on this dimension Cole, Mailath, and Postlewaite (2001a) is closest to our setting. In such a framework Cole, Mailath and Postlewaite demonstrate the existence of an equilibrium allocation that induces efficient investments as well as allocations that yield inefficiencies. When the numbers of buyers and sellers are discrete they are able to uniquely select an equilibrium allocation of the matches' surplus

⁵Notice that Ramey and Watson (1996) also consider how matching frictions can alleviate the inefficiencies due to the hold-up problem in the presence of incomplete contracts and match specific investments in an ongoing repeated relationship. See also Ramey and Watson (1997) for a related result.

yielding efficient investments via a condition defined as ‘double-overlapping’. This condition requires the presence of at least two buyers (or two sellers) with identical innate characteristics; it implies the existence of a perfect substitute for each buyer and each seller in the match. In other words, competition does not occur among individuals that are fully heterogeneous with respect to their innate characteristics. In this case, both sides to a match obtain exactly their outside option and, at the same time, their most favorable share of the surplus hence efficiency is promoted. In the absence of double-overlapping — therefore when competing individuals are fully heterogeneous — equilibrium investments may not be efficient since at least one of the parties to a match is not obtaining the most favorable share of the match surplus. This creates room for equilibria with under-investments though Cole, Mailath, and Postlewaite (2001a) show that, even in the latter case, there exists a sharing rule of the surplus that leads to efficient investments.

Our analysis differs from Cole, Mailath, and Postlewaite (2001a) in that we do not use cooperative game concepts and matching is through a non-cooperative Bertrand competition game. With a cooperative game approach, the mechanics of the competition game are not specified. This gives generality but some elements of strategic structure remain unspecified. With a non-cooperative approach, the structure is fully specified and we are able to analyze the extent of the inefficiency under an “equilibrium” sharing rule. Each seller’s outside option is binding for any value of the buyers’ and sellers’ innate characteristics. However a buyer’s outside option is never binding although buyers do obtain their most preferred share of the match’s surplus. We thus choose a particular model of the competition among fully heterogeneous individuals and thanks to this specific extensive form we are able to provide a bound on the overall inefficiency that arises because of the sellers’ underinvestments.

Finally, de Meza and Lockwood (1998) and Chatterjee and Chiu (1999) also analyze a matching environment with transferable utility in which both sides of the market can undertake match specific investments but focus on a setup that delivers inefficient investments. As a result the presence of asset ownership may enhance welfare (as in Grossman and Hart 1986). In particular, de Meza and Lockwood (1998)

consider a repeated production framework and focus on whether one would observe asset trading before or after investment and match formation. Chatterjee and Chiu (1999), on the other hand, analyze a setup in which, as in our case, trade occurs only once. The inefficiency takes the form of the choice of general investments when specific ones would be efficient and arise from the way surplus is shared by the parties to a match when the short side of the market undertakes the investments. They focus on the (possibly adverse) efficiency enhancing effect of ownership of assets. In our setting we abstract from any efficiency enhancing role of asset ownership.

3. The Framework

We consider a simple matching model: S buyers match with T sellers, we assume that the number of buyers is higher than the number of sellers $S > T$. Each seller is assumed to match only with one buyer. Buyers and sellers are labelled, respectively, $s = 1, \dots, S$ and $t = 1, \dots, T$. Both buyers and sellers can make match specific investments, denoted respectively x_s and y_t , incurring costs $C(x_s)$ respectively $C(y_t)$.⁶ The cost function $C(\cdot)$ is strictly convex and $C(0) = 0$. The surplus of each match is then a function of the quality of the buyer σ and the seller τ involved in the match: $v(\sigma, \tau)$. Each buyer's quality is itself a function of the buyer's innate ability, indexed by his identity s , and the buyer's specific investment x_s : $\sigma(s, x_s)$. In the same way, we assume that each seller's quality is a function of the seller's innate ability, indexed by her identity t , and the seller's specific investment y_t : $\tau(t, y_t)$.

We assume *complementarity* of the qualities of the buyer and the seller involved in a match. In other words, the higher is the quality of the buyer and the seller the higher is the surplus generated by the match:⁷ $v_1(\sigma, \tau) > 0$, $v_2(\sigma, \tau) > 0$. Further, the marginal surplus generated by a higher quality of the buyer or of the seller in

⁶For simplicity we take both cost functions to be identical, none of our results depending on this assumption. If the cost functions were type specific we would require the marginal costs to increase with the identity of the buyer or the seller.

⁷For convenience we denote with $v_l(\cdot, \cdot)$ the partial derivative of the surplus function $v(\cdot, \cdot)$ with respect to the l -th argument and with $v_{lk}(\cdot, \cdot)$ the cross-partial derivative with respect to the l -th and k -th argument or the second-partial derivatives if $l = k$. We use the same notation for the functions $\sigma(\cdot, \cdot)$ and $\tau(\cdot, \cdot)$ defined above.

the match increases with the quality of the partner: $v_{12}(\sigma, \tau) > 0$. We also assume that the quality of the buyer depends negatively on the buyer's innate ability s , $\sigma_1(s, x_s) < 0$ (so that buyer $s = 1$ is the highest ability buyer) and positively on the buyer's specific investment x_s : $\sigma_2(s, x_s) > 0$. Similarly, the quality of a seller depends negatively on the seller's innate ability t , $\tau_1(t, y_t) < 0$, (seller $t = 1$ is the highest ability seller) and positively on the seller's investment y_t : $\tau_2(t, y_t) > 0$. Finally we assume that the quality of both the buyers and the sellers satisfy a *single crossing condition* requiring that the marginal productivity of both buyers and sellers investments decreases in their innate ability index: $\sigma_{12}(s, x_s) < 0$ and $\tau_{12}(t, y_t) < 0$.

The combination of the assumption of complementarity and the single crossing condition gives a particular meaning to the term specific investments we used for x_s and y_t . Indeed, in our setting the investments x_s and y_t have a use and value in matches other than (s, t) ; however, these values decrease with the identity of the partner implying that at least one component of this value is specific to the match in question, since we consider a discrete number of buyers and sellers.

We also assume that the surplus of each match is concave in the buyers and sellers quality — $v_{11}(\sigma, \tau) < 0$, $v_{22}(\sigma, \tau) < 0$ — and that the quality of both sellers and buyers exhibit decreasing marginal returns in their investments: $\sigma_{22}(\sigma, \tau) < 0$ and $\tau_{22}(\sigma, \tau) < 0$.⁸

We analyze different specifications of our general framework.

We first characterize (Section 4 below) the equilibrium of the Bertrand competition game for given vectors of sellers' and buyers' qualities.

We then move (Section 5 below) to the analysis of the buyers' investment choice in a model in which only the buyers choose ex-ante match specific investments x_s that determine the quality of each buyer $\sigma(s, x_s)$ while sellers are of exogenously given qualities: $\tau(t)$.

⁸As established in Milgrom and Roberts (1990), Milgrom and Roberts (1994) and Edlin and Shannon (1998) our results can be derived with much weaker assumptions on the smoothness and concavity of the surplus function $v(\cdot, \cdot)$ and the two quality functions $\sigma(\cdot, \cdot)$ and $\tau(\cdot, \cdot)$ in the two investments x_s and y_t .

We conclude (Section 6 and 7 below) with the analysis of the sellers' investment choice in the model in which only sellers choose ex-ante match specific investments y_t that determine each seller t 's quality $\tau(t, y_t)$ while buyers are of exogenously given quality $\sigma(s)$.

The case in which both sellers and buyers undertake ex-ante investments is briefly discussed in the conclusions.

We assume the following extensive forms of the Bertrand competition game in which the T sellers and the S buyers engage. Buyers Bertrand compete for sellers. All buyers simultaneously and independently submit bids to the T sellers. Notice that we allow buyers to submit bids to more than one, possibly all sellers. Each seller observes the bids she received and decides which offer to accept. We assume that this decision is taken sequentially in the order of a given permutation (t_1, \dots, t_T) of the vector of sellers' identities $(1, \dots, T)$. In other words, the seller labelled t_1 decides first which bid to accept. This commits the buyer selected to a match with seller t_1 and automatically withdraws all bids this buyer made to the other sellers. All other sellers and buyers observe this decision and then seller t_2 decides which bid to accept. This process is repeated until seller t_T decides which bid to accept. Notice that since $S > T$ even seller t_T , the last seller to decide, can potentially choose among multiple bids.

In Sections 5 and 6 below we focus mainly on the case in which sellers choose their bids in the decreasing order of their identity (innate ability): $t_n = n$, for all $n = 1, \dots, T$. We endogenize this choice in Subsection 7.1 below.

We look for the *trembling-hand-perfect* equilibria of our model so as to rule out equilibria in which (unsuccessful) bids exceed buyers' valuations. In the construction of the trembling-hand-perfect equilibrium we allow buyers, when submitting a bid, to state that they are prepared to bid more if this becomes necessary. We then restrict the totally mixed strategy of each seller to be such that each seller selects bids starting with a higher-order probability on the highest bids and allocates a lower-order probability of being selected on a bid submitted by a buyer that did not specify

such a proviso.⁹

The logic behind this additional restriction derives from the observation that in the extensive form of the Bertrand game there exists an asymmetry between the timing of buyers's bids (they are all simultaneously submitted at the beginning of the Bertrand competition subgame) and the timing of each seller's choice of the bid to accept (sellers choose their most preferred bid sequentially in a given order). This implies that while in equilibrium it is possible that a seller's choice between two identical bids is uniquely determined this is not any more true following a deviation of a buyer whose bid in equilibrium is selected by a seller who gets to choose her most preferred bid at an earlier stage of the subgame. To prevent sellers from deviating when choosing among identical bids following a buyer's deviation — that possibly does not even affect the equilibrium bids submitted to the seller in question — we chose to modify the extensive form in the way described above.

4. Bertrand Competition

We now proceed to characterize the equilibria of the model described in Section 3 above solving it backwards. In particular we start from the characterization of the equilibrium of the Bertrand competition subgame. In doing so we take the investments and hence the qualities of both sellers and buyers for given.

To simplify the analysis below let τ_1 be the quality of seller t_1 that, as described in Section 3 above, is the first seller to choose her most preferred bid in the Bertrand competition subgame. In a similar way, denote τ_n the quality of seller t_n , $n = 1, \dots, T$, that is the n -th seller to choose her most preferred bid. The vector of sellers' qualities is then (τ_1, \dots, τ_T) .

We first identify an efficiency property of any equilibrium of the Bertrand competition subgame. All the equilibria of the Bertrand competition subgame exhibit *positive assortative matching*. In other words, for given investments, matches are

⁹This modification of the extensive form is equivalent to a Bertrand competition model in which there exists an indivisible smallest possible unit of a bid (a penny) so that each buyer can break any tie by bidding one penny more than his opponent if he wishes to do so.

efficient: the buyer characterized by the k -th highest quality matches with the seller characterized by the k -th highest quality.

Lemma 1. *Every equilibrium of the Bertrand competition subgame is such that every pair of equilibrium matches (σ', τ_i) and (σ'', τ_j) , $i, j \in \{1, \dots, T\}$ satisfies the property: If $\tau_i > \tau_j$ then $\sigma' > \sigma''$.*

Proof: Assume by way of contradiction that the equilibrium matches are not assortative. In other words, there exist a pair of equilibrium matches (σ'', τ_i) and (σ', τ_j) such that $\tau_i > \tau_j$, and $\sigma' > \sigma''$. Denote $b(\tau_i)$, respectively $b(\tau_j)$, the bids that in equilibrium the seller of quality τ_i , respectively of quality τ_j , accepts.

Consider first the match (σ'', τ_i) . For this match to occur in equilibrium we need that it is not convenient for the buyer of quality σ'' to match with the seller of quality τ_j rather than τ_i . If buyer σ'' deviates and does not submit a bid that will be selected by seller τ_i then two situations may occur depending on whether the seller of quality τ_i chooses her bid before, ($i < j$), or after ($i > j$), the seller of quality τ_j . In particular if τ_i chooses her bid before τ_j then following the deviation of the buyer of quality σ'' a different buyer will be matched with seller τ_i . Then the competition for the seller of quality τ_{i+1} will be won either by the same buyer as in the absence of the deviation or, if that buyer has already been matched, by another buyer who now would not be bidding for subsequent sellers. Repeating this argument for subsequent sellers we conclude that when following a deviation by buyer σ'' it is the turn of the seller of quality τ_j to choose her most preferred bid the set of unmatched buyers, excluding buyer σ'' , is depleted of exactly one buyer, if compared with the set of unmatched buyers when in equilibrium the seller of quality τ_j chooses her most preferred bid. Hence the maximum bids of these buyers $\hat{b}(\tau_j)$ cannot be higher than the equilibrium bid $b(\tau_j)$ of the buyer of quality σ' : $\hat{b}(\tau_j) \leq b(\tau_j)$.¹⁰

¹⁰Notice that we can conclude that following a deviation by buyer σ'' the bid accepted by seller τ_j is not higher than $b(\tau_j)$ since — as discussed in Section 3 above — we allow buyers to specify in their bid that they are willing to increase such a bid if necessary. Moreover we restrict the totally mixed strategy used by each seller so as to put higher order probabilities on the bids that contain this

Therefore for (σ'', τ_i) to be an equilibrium match we need that

$$v(\sigma'', \tau_i) - b(\tau_i) \geq v(\sigma'', \tau_j) - \hat{b}(\tau_j) \quad (1)$$

or given that, as argued above, $\hat{b}(\tau_j) \leq b(\tau_j)$ we need that the following necessary condition is satisfied:

$$v(\sigma'', \tau_i) - b(\tau_i) \geq v(\sigma'', \tau_j) - b(\tau_j) \quad (2)$$

Alternatively if τ_i chooses her bid after τ_j then for (σ'', τ_i) to be an equilibrium match we need that buyer σ'' does not find convenient to deviate and outbid the buyer of quality σ' by submitting bid $b(\tau_j)$. This equilibrium condition therefore coincides with (2) above.

Consider now the equilibrium match (σ', τ_j) . For this match to occur in equilibrium we need that the buyer of quality σ' does not want to deviate and be matched with the seller of quality τ_i rather than τ_j . As discussed above, depending on whether the seller of quality τ_j chooses her bid before, ($j < i$), or after, ($j > i$), the seller of quality τ_i , the following is a necessary condition for (σ', τ_j) to be an equilibrium match:

$$v(\sigma', \tau_j) - b(\tau_j) \geq v(\sigma', \tau_i) - b(\tau_i). \quad (3)$$

The inequalities (2) and (3) imply:

$$v(\sigma'', \tau_i) + v(\sigma', \tau_j) \geq v(\sigma', \tau_i) + v(\sigma'', \tau_j). \quad (4)$$

Condition (4) contradicts the complementarity assumption $v_{12}(\sigma, \tau) > 0$. ■

Notice that, as argued in Section 5 and 6 below, Lemma 1 does not imply that the

proviso. In the absence of these restrictions it is possible to envisage a situation in which following a deviation by buyer σ'' the sellers that select their bid after seller τ_i and before seller τ_j may no longer choose among equal bids the one submitted by the buyer with the highest willingness to pay. The result is then that the bid accepted by seller τ_j following a deviation might actually be higher than $b(\tau_j)$. Notice that this problem disappears if we assume that there exists a smallest indivisible unit of a bid (see also Footnote 9 above).

order of sellers' qualities, which are endogenously determined by sellers' investments, coincides with the order of sellers' identities (innate abilities).

Using Lemma 1 above we can now label buyers' qualities in a way that is consistent with the way sellers' qualities are labelled. Indeed, Lemma 1 defines an equilibrium relationship between the quality of each buyer and the quality of each seller. We can therefore denote σ_n , $n = 1, \dots, T$ the quality of the buyer that in equilibrium matches with seller τ_n . Furthermore, we denote $\sigma_{T+1}, \dots, \sigma_S$ the qualities of the buyers that in equilibrium are not matched with any seller and assume that these qualities are ordered so that $\sigma_i > \sigma_{i+1}$ for all $i = T + 1, \dots, S - 1$.

Consider now stage t of the Bertrand competition subgame characterized by the fact that the seller of quality τ_t chooses her most preferred bid. The buyers that are still unmatched at this stage of the subgame are the ones with qualities $\sigma_t, \sigma_{t+1}, \dots, \sigma_S$. We define the *runner-up* buyer to the seller of quality τ_t to be the buyer, among the ones with qualities $\sigma_{t+1}, \dots, \sigma_S$, who has the highest willingness to pay for a match with seller τ_t . We denote this buyer $r(t)$ and his quality $\sigma_{r(t)}$. Clearly $r(t) > t$.

This definition can be used recursively so as to define the runner-up buyer to the seller that is matched in equilibrium with the runner-up buyer to the seller of quality τ_t . We denote this buyer $r^2(t) = r(r(t))$ and his quality $\sigma_{r^2(t)}$: $r^2(t) > r(t) > t$. In an analogous way we can then denote $r^k(t) = r(r^{k-1}(t))$ for every $k = 1, \dots, \rho_t$ where $r^k(t) > r^{k-1}(t)$, $r^1(t) = r(t)$ and $\sigma_{r^{\rho_t}(t)}$ is the quality of the last buyers in the chain of runner-ups to the seller of quality τ_t .

We have now all the elements to provide a characterization of the equilibrium of the Bertrand competition subgame. In particular we first identify the runner-up buyer to every seller and the difference equation satisfied by the equilibrium payoffs to all sellers and buyers. This is done in the following lemma.

Lemma 2. *The runner-up buyer to the seller of quality τ_t , $t = 1, \dots, T$, is the buyer*

of quality $\sigma_{r(t)}$ such that:

$$\sigma_{r(t)} = \max \{ \sigma_i \mid i = t + 1, \dots, S \text{ and } \sigma_i \leq \sigma_t \}. \quad (5)$$

Further the equilibrium payoffs to each buyer, $\pi_{\sigma_t}^B$ and each seller, $\pi_{\tau_t}^S$, are such that for every $t = 1, \dots, T$:

$$\pi_{\sigma_t}^B = [v(\sigma_t, \tau_t) - v(\sigma_{r(t)}, \tau_t)] + \pi_{\sigma_{r(t)}}^B \quad (6)$$

$$\pi_{\tau_t}^S = v(\sigma_{r(t)}, \tau_t) - \pi_{\sigma_{r(t)}}^B \quad (7)$$

and for every $i = T + 1, \dots, S$:

$$\pi_{\sigma_i}^B = 0 \quad (8)$$

We present the formal proof of this result in the Appendix. Notice however that equation (5) identifies the runner-up buyer of the seller of quality τ_t as the buyer — other than the one of quality σ_t that in equilibrium matches with seller τ_t — which has the highest quality among the buyers with quality lower than σ_t that are still unmatched at stage t of the Bertrand competition subgame. For any seller of quality τ_t it is then possible to construct a chain of runner-up buyers: each one the runner-up buyer to the seller that in equilibrium is matched with the runner-up buyer that is ahead in the chain. Equation (5) implies that for every seller the last buyer in the chain of runner-up buyers is the buyer of quality σ_{T+1} . This is the highest quality buyer among the ones that in equilibrium do not match with any seller. In other words every chain of runner-up buyers has at least one buyer in common.

Given that buyers Bertrand compete for sellers, each seller will not be able to capture all the match surplus but only her outside option that is determined by the willingness to pay of the runner-up buyer to the seller. This willingness to pay is the difference between the surplus of the match between the runner-up buyer and the seller in question and the payoff the runner-up buyer obtains in equilibrium if he is not successful in his bid to the seller: the difference equation in (7). Given that the quality of the runner-up buyer is lower than the quality of the buyer the seller

is matched with in equilibrium the share of the surplus each seller is able to capture does not coincide with the entire surplus of the match. The payoff to each buyer is then the difference between the surplus of the match and the runner-up buyer's bid: the difference equation in (6).

The characterization of the equilibrium of the Bertrand competition subgame is summarized in the following proposition.

Proposition 1. *For any given vector of sellers' qualities (τ_1, \dots, τ_T) and corresponding vector of buyers' qualities $(\sigma_1, \dots, \sigma_S)$, the unique equilibrium of the Bertrand competition subgame is such that every pair of equilibrium matches (σ_i, τ_i) and (σ_j, τ_j) , $i, j \in \{1, \dots, T\}$, is such that:*

$$\text{If } \tau_i > \tau_j \quad \text{then} \quad \sigma_i > \sigma_j. \quad (9)$$

Further, the equilibrium shares of the match surplus that each buyer of quality σ_t and each seller of quality τ_t , $t = 1, \dots, T$, receive are such that:

$$\begin{aligned} \pi_{\sigma_t}^B = & [v(\sigma_t, \tau_t) - v(\sigma_{r(t)}, \tau_t)] + \\ & + \sum_{k=1}^{\rho_t} [v(\sigma_{r^k(t)}, \tau_{r^k(t)}) - v(\sigma_{r^{k+1}(t)}, \tau_{r^k(t)})] \end{aligned} \quad (10)$$

$$\pi_{\tau_t}^S = v(\sigma_{r(t)}, \tau_t) - \sum_{k=1}^{\rho_t} [v(\sigma_{r^k(t)}, \tau_{r^k(t)}) - v(\sigma_{r^{k+1}(t)}, \tau_{r^k(t)})] \quad (11)$$

where $r^{\rho_t}(t) = T + 1$ and $v(\sigma_{r^{\rho_t}(t)}, \tau_{r^{\rho_t}(t)}) = v(\sigma_{r^{\rho_t+1}(t)}, \tau_{r^{\rho_t}(t)}) = 0$.

Proof: Condition (9) is nothing but a restatement of Lemma 1. The proof of (10) and (11) follows directly from Lemma 2. In particular, solving recursively (6), using (8), we obtain (10); then substituting (10) into (7) we obtain (11). ■

We now analyze the unique equilibrium of the Bertrand competition subgame in the case in which the order in which sellers select their most preferred bid is the decreasing order of their qualities: $\tau_1 > \dots > \tau_T$ and $\sigma_1 > \dots > \sigma_S$. From Lemma 2

— condition (5) — this also implies that the runner-up buyer to the seller of quality τ_t is the buyer of quality σ_{t+1} for every $t = 1, \dots, T$. The following proposition characterizes the equilibrium of the Bertrand competition subgame in this case.

Proposition 2. *For any given ordered vector of sellers' qualities (τ_1, \dots, τ_T) so that $\tau_1 > \dots > \tau_T$ and corresponding vector of buyers' qualities $(\sigma_1, \dots, \sigma_S)$ the unique equilibrium of the Bertrand competition subgame is such that the equilibrium matches are (σ_k, τ_k) , $k = 1, \dots, T$ and the shares of the match surplus that each buyer of quality σ_t and each seller of quality τ_t receive are such that:*

$$\pi_{\sigma_t}^B = \sum_{h=t}^T [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)] \quad (12)$$

$$\pi_{\tau_t}^S = v(\sigma_{t+1}, \tau_t) - \sum_{h=t+1}^T [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)] \quad (13)$$

Proof: This result follows directly from Lemma 1, Lemma 2 and Proposition 1 above. In particular, (5) implies that when (τ_1, \dots, τ_T) and $(\sigma_1, \dots, \sigma_S)$ are ordered vectors of qualities $\sigma_{r(t)} = \sigma_{t+1}$ for every $t = 1, \dots, T$. Then substituting the identity of the runner-up buyer in (10) and (11) we obtain (12) and (13). ■

The main difference between Proposition 2 and of Proposition 1 can be described as follows. Consider the subgame in which the seller of quality τ_t chooses among her bids and let (τ_1, \dots, τ_T) be an ordered vector of qualities as in Proposition 2. This implies that $\sigma_t > \sigma_{t+1} > \sigma_{t+2}$. The runner-up buyer to the seller with quality τ_t is then the buyer of quality σ_{t+1} and the willingness to pay of this buyer (hence the share of the surplus accruing to seller τ_t) is, from (7) above:

$$v(\sigma_{t+1}, \tau_t) - \pi_{\sigma_{t+1}}^B. \quad (14)$$

Notice further that since the runner-up buyer to seller τ_{t+1} is σ_{t+2} from (6) above the

payoff to the buyer of quality σ_{t+1} is:

$$\pi_{\sigma_{t+1}}^B = v(\sigma_{t+1}, \tau_{t+1}) - v(\sigma_{t+2}, \tau_{t+1}) + \pi_{\sigma_{t+2}}^B. \quad (15)$$

Substituting (15) into (14) we obtain that the willingness to pay of the runner-up buyer σ_{t+1} is then:

$$v(\sigma_{t+1}, \tau_t) - v(\sigma_{t+1}, \tau_{t+1}) + v(\sigma_{t+2}, \tau_{t+1}) - \pi_{\sigma_{t+2}}^B. \quad (16)$$

Consider now a new vector of sellers qualities $(\tau_1, \dots, \tau'_{t-1}, \tau_t, \tau'_{t+1}, \dots, \tau_T)$ where the qualities τ_i for every i different from $t-1$ and $t+1$ are the same as the ones in the ordered vector (τ_1, \dots, τ_T) . Assume that $\tau'_{t-1} = \tau_{t+1} < \tau_t$ and $\tau'_{t+1} = \tau_{t-1} > \tau_t$. This assumption implies that the vector of buyers' qualities $(\sigma'_1, \dots, \sigma'_S)$ differs from the ordered vector of buyers qualities $(\sigma_1, \dots, \sigma_S)$ only in its $(t-1)$ -th and $(t+1)$ -th components that are such that: $\sigma'_{t-1} = \sigma_{t+1} < \sigma_t$ and $\sigma'_{t+1} = \sigma_{t-1} > \sigma_t$. From (5) above we have that the runner-up buyer for seller τ_t is now buyer σ_{t+2} and the willingness to pay of this buyer is:

$$v(\sigma_{t+2}, \tau_t) - \pi_{\sigma_{t+2}}^B. \quad (17)$$

Comparing (16) with (17) we obtain, by the complementarity assumption $v_{12}(\sigma, \tau) > 0$, that

$$v(\sigma_{t+1}, \tau_t) - v(\sigma_{t+1}, \tau_{t+1}) + v(\sigma_{t+2}, \tau_{t+1}) > v(\sigma_{t+2}, \tau_t).$$

In other words, the willingness to pay of the runner-up buyer to seller τ_t in the case considered in Proposition 2 is strictly greater than the willingness to pay of the runner-up buyer to seller τ_t in the special case of Proposition 1 we just considered. The reason is that in the latter case there is one less buyer σ_{t+1} to actively compete for the match with seller τ_t . This comparison is generalized in the following proposition proved in the Appendix.

Proposition 3. *Let (τ_1, \dots, τ_T) be an ordered vector of sellers qualities so that $\tau_1 > \dots > \tau_T$ and $(\tau'_1, \dots, \tau'_T)$ be any permutation of the vector (τ_1, \dots, τ_T) with the same*

t -th element: $\tau'_t = \tau_t$ such that there exists an $i < t$ that permutes into a τ'_j , ($\tau_i = \tau'_j$), with $j > t$. Denote $(\sigma_1, \dots, \sigma_T)$ and $(\sigma'_1, \dots, \sigma'_T)$ the corresponding vectors of buyers' qualities. Then seller τ_t 's payoff, as in (13), is greater than seller τ'_t 's payoff, as in (11):

$$\begin{aligned} v(\sigma_{t+1}, \tau_t) &- \sum_{h=t+1}^T [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)] > \\ &> v(\sigma'_{r(t)}, \tau'_t) - \sum_{k=1}^{\rho'_t} \left[v(\sigma'_{r^k(t)}, \tau'_{r^k(t)}) - v(\sigma'_{r^{k+1}(t)}, \tau'_{r^{k+1}(t)}) \right] \end{aligned} \quad (18)$$

Proposition 3 allow us to conclude that when sellers select their preferred bid in the decreasing order of their qualities competition among buyers for each match is at its peak.¹¹ This is apparent when we consider the case in which the order in which sellers select their most preferred bid is the increasing order of their qualities: $\tau_1 < \dots < \tau_T$. In this case, according to (5) above, the runner-up buyer to each seller has quality σ_{T+1} . This implies that the payoff to each seller $t = 1, \dots, T$ is:

$$\pi_{\tau_t}^S = v(\sigma_{T+1}, \tau_t) \quad (19)$$

In this case only two buyers — the buyer of quality σ_t and the buyer of quality σ_{T+1} — actively compete for the match with seller τ_t and sellers' payoffs are at their minimum.

Given that in our analysis we stress the role of competition in solving the inefficiencies due to match-specific investments in what follows we mainly focus on the case in which sellers choose their most preferred bid in the decreasing order of their innate ability. Notice that this does not necessarily mean that sellers choose their most preferred bid in the decreasing order of their qualities $\tau_1 > \dots > \tau_T$ and hence competition among buyers is at its peak. Indeed, sellers' qualities are endogenously determined in the analysis that follows. We come back to this point in Subsection

¹¹Notice that trembling-hand-perfection implies that all unmatched buyers with a strictly positive willingness to pay for the match with a given seller submit their bids in equilibrium.

7.1 below where we endogenize the order in which sellers choose their most preferred bid.

We conclude this section by observing that from Proposition 1 above, the buyer's equilibrium payoff $\pi_{\sigma_t}^B$ is the sum of the social surplus produced by the equilibrium match $v(\sigma_t, \tau_t)$ and an expression \mathcal{B}_{σ_t} that does not depend on the quality σ_t of the buyer involved in the match. In particular, this implies that \mathcal{B}_{σ_t} does not depend on the match-specific investment of the buyer of quality σ_t :

$$\pi_{\sigma_t}^B = v(\sigma_t, \tau_t) + \mathcal{B}_{\sigma_t}. \quad (20)$$

Moreover, from (11), each seller's equilibrium payoff $\pi_{\tau_t}^S$ is also the sum of the surplus generated by the inefficient (if it occurs) match of the seller of quality τ_t with the runner-up buyer of quality $\sigma_{r(t)}$ and an expression \mathcal{S}_{τ_t} that does not depend on the match-specific investment of the seller of quality τ_t :

$$\pi_{\tau_t}^S = v(\sigma_{r(t)}, \tau_t) + \mathcal{S}_{\tau_t}. \quad (21)$$

Of course when sellers select their bids in the decreasing order of their qualities the runner-up buyer to seller t is the buyer of quality σ_{t+1} , as from (5) above. Therefore equation (21) becomes:

$$\pi_{\tau_t}^S = v(\sigma_{t+1}, \tau_t) + \mathcal{S}_{\tau_t}. \quad (22)$$

These conditions play a crucial role when we analyze the efficiency of the investment choices of both buyers and sellers.

5. Buyers' Investments

In this section we analyze the model under the assumption that the quality of sellers is exogenously given $\tau(t)$ while the quality of buyers depends on both the buyers' identity (innate ability) and their match specific investments $\sigma(s, x_s)$.

We consider first the case in which sellers choose their preferred bids in the decreasing order of their innate abilities. In this contest since sellers' qualities are

exogenously determined this assumption coincides with the assumption that sellers choose their preferred bid in the decreasing order of their qualities $\tau_1 > \dots > \tau_T$. Hence, Proposition 2 provides the characterization of the unique equilibrium of the Bertrand competition subgame in this case.

We proceed to characterize the equilibrium of the buyers' investment game. We first show that an equilibrium of this simultaneous move investment game always exist and that this equilibrium is efficient: the order of the induced qualities $\sigma(s, x_s)$, $s = 1, \dots, S$, coincides with the order of the buyers' identities s , $s = 1, \dots, S$. We then show that an inefficiency may arise, depending on the distribution of sellers' qualities and buyers' innate abilities. This inefficiency takes the form of additional inefficient equilibria, such that the order of the buyers' identities differs from the order of their induced qualities.

Notice first that each buyer's investment choice is efficient given the equilibrium match the buyer is involved in. Indeed, the Bertrand competition game will make each buyer residual claimant of the surplus produced in his equilibrium match. Therefore, the buyer is able to appropriate the marginal returns from his investment and hence his investment choice is efficient given the equilibrium match.

Assume that the equilibrium match is the one between the s buyer and the t seller, from equation (20) buyer s 's optimal investment choice $x_s(t)$ is the solution to the following problem:

$$x_s(t) = \operatorname{argmax}_x \pi_{\sigma(s,x)}^B - C(x) = v(\sigma(s, x), \tau_t) - \mathcal{B}_{\sigma(s,x)} - C(x). \quad (23)$$

This investment choice is defined by the following necessary and sufficient first order conditions of problem (23):

$$v_1(\sigma(s, x_s(t)), \tau_t) \sigma_2(s, x_s(t)) = C'(x_s(t)). \quad (24)$$

where $C'(\cdot)$ is the first derivative of the cost function $C(\cdot)$.

Notice that (24) follows from the fact that $\mathcal{B}_{\sigma(s,x)}$ does not depend on buyer s 's

quality $\sigma(s, x)$, and hence on buyer s 's match specific investment x . The following two lemmas derive the properties of buyer s 's investment choice $x_s(t)$ and his quality $\sigma(s, x_s(t))$.

Lemma 3. *For any given equilibrium match $(\sigma(s, x_s(t)), \tau_t)$ buyer s 's investment choice $x_s(t)$, as defined in (24), is constrained efficient.*

Proof: Notice first that if a central planner is constrained to choose the match between buyer s and seller t buyer s 's constrained efficient investment is the solution to the following problem:

$$x^*(s, t) = \underset{x}{\operatorname{argmax}} v(\sigma(s, x), \tau_t) - C(x). \quad (25)$$

This investment $x^*(s, t)$ is defined by the following necessary and sufficient first order conditions of problem (25):

$$v_1(\sigma(s, x^*(s, t)), \tau_t) \sigma_2(s, x^*(s, t)) = C'(x^*(s, t)). \quad (26)$$

The result then follows from the observation that the definition of the constrained efficient investment $x^*(s, t)$, equation (26), coincides with the definition of buyer s 's optimal investment $x_s(t)$, equation (24) above. ■

Lemma 4. *For any given equilibrium match $(\sigma(s, x_s(t)), \tau_t)$ buyer s 's optimally chosen quality $\sigma(s, x_s(t))$ decreases both in the buyer's identity s and in the seller identity t :*

$$\frac{d\sigma(s, x_s(t))}{ds} < 0, \quad \frac{d\sigma(s, x_s(t))}{dt} < 0.$$

Proof: The result follows from condition (24) that implies:

$$\frac{d\sigma(s, x_s(t))}{ds} = \frac{\sigma_1 v_1 \sigma_{22} - \sigma_1 C'' - v_1 v_2 \sigma_{12}}{v_{11} (\sigma_2)^2 + v_1 \sigma_{22} - C''} < 0,$$

and

$$\frac{d\sigma(s, x_s(t))}{dt} = \frac{v_{12} (\sigma_2)^2}{v_{11} (\sigma_2)^2 + v_1 \sigma_{22} - C''} < 0,$$

where the functions σ_h and σ_{hk} , $h, k \in \{1, 2\}$, are computed at $(s, x_s(t))$; the functions v_h and v_{hk} , $h, k \in \{1, 2\}$, are computed at $(\sigma(s, x_s(t)), \tau_t)$ and the second derivative of the cost function C'' is the second derivative of the cost function $C(\cdot)$ computed at $x_s(t)$. ■

We define now an equilibrium of the buyers' investment game. Let (s_1, \dots, s_S) denote a permutation of the vector of buyers' identities $(1, \dots, S)$. An equilibrium of the buyers' investment game is then a vector of investment choices $x_{s_i}(i)$, as defined in (24) above, such that the resulting buyers' qualities have the same order as the identity of the associated sellers:

$$\sigma(s_i, x_{s_i}(i)) = \sigma_i < \sigma(s_{i-1}, x_{s_{i-1}}(i-1)) = \sigma_{i-1} \quad \forall i = 2, \dots, S, \quad (27)$$

where σ_i denotes the i -th element of the equilibrium ordered vector of qualities $(\sigma_1, \dots, \sigma_S)$.¹²

Notice that this equilibrium definition allows for the order of buyers' identities to differ from the order of their qualities and therefore from the order of the identities of the sellers each buyer is matched with.

We can now proceed to show the existence of the efficient equilibrium of the buyer investment game. This is the equilibrium characterized by the coincidence of the order of buyers' identities and the order of their qualities. From Lemma 1 the efficient equilibrium matches are $(\sigma(t, x_t(t)), \tau_t)$, $t = 1, \dots, T$.

Proposition 4. *The equilibrium of the buyers' investment game characterized by $s_i = i$, $i = 1, \dots, S$ always exists and is efficient.*

The formal proof of this result is presented in the Appendix. However the intuitive argument behind this proof is simple to describe. The payoff to buyer i , $\pi_i^B(\sigma) -$

¹²Recall that since $\tau_1 > \dots > \tau_T$ Lemma 1 and the notation defined in Section 4 above imply that $\sigma_1 > \dots > \sigma_S$.

$C(x(i, \sigma))$, changes expression as buyer i increases his investment so as to improve his quality and match with a higher quality seller.¹³ This payoff however is continuous at any point, such as σ_{i-1} , in which in the continuation Bertrand game the buyer matches with a different seller, but has a kink at such points.¹⁴

However, if the equilibrium considered is the efficient one — $s_i = i$ for every $i = 1, \dots, S$ — the payoff to buyer i is monotonic decreasing in any interval to the right of the $(\sigma_{i+1}, \sigma_{i-1})$ and increasing in any interval to the left. Therefore, this payoff has a unique global maximum. Hence buyer i has no incentive to deviate and change his investment choice.

If instead we consider an inefficient equilibrium — an equilibrium where s_1, \dots, s_S differs from $1, \dots, S$ — then the payoff to buyer i is still continuous at any point, such as $\sigma(s_i, x_{s_i}(i))$, in which in the continuation Bertrand game the buyer gets matched with a different seller. However, this payoff is not any more monotonic decreasing in any interval to the right of the $(\sigma(s_{i+1}, x_{s_{i+1}}(i+1)), \sigma(s_{i-1}, x_{s_{i-1}}(i-1)))$ and increasing in any interval to the left. In particular, this payoff is increasing at least in the right neighborhood of the switching points $\sigma(s_h, x_{s_h}(h))$ for $h = 1, \dots, i-1$ and decreasing in the left neighborhood of the switching points $\sigma(s_k, x_{s_k}(k))$ for $k = i+1, \dots, N$.

This implies that depending on the values of parameters these inefficient equilibria may or may not exist. We show below that for given sellers' qualities it is possible to construct inefficient equilibria if two buyers' qualities are close enough. Alternatively, for given buyers' qualities inefficient equilibria do not exist if the sellers qualities are close enough.

Proposition 5. *Given any ordered vector of sellers' qualities (τ_1, \dots, τ_T) , it is possible to construct an inefficient equilibrium of the buyers' investment game such that there exists at least an i that satisfies $s_i < s_{i-1}$.*

¹³The level of investment $x(i, \sigma)$ is defined, as in the Appendix: $\sigma(i, x) \equiv \sigma$.

¹⁴Indeed, from (A.20) and (A.21) we get that $\frac{\partial[\pi_i^B(\sigma_{i-1}^-) - C(x(i, \sigma_{i-1}^-))]}{\partial \sigma} = v_1(\sigma_{i-1}, \tau_i) - \frac{C'(x(i, \sigma_{i-1}^-))}{\sigma_2(i, x(i, \sigma_{i-1}^-))}$ and $\frac{\partial[\pi_i^B(\sigma_{i-1}^+) - C(x(i, \sigma_{i-1}^+))]}{\partial \sigma} = v_1(\sigma_{i-1}, \tau_{i-1}) - \frac{C'(x(i, \sigma_{i-1}^+))}{\sigma_2(i, x(i, \sigma_{i-1}^+))}$. Therefore, from $v_{12}(\sigma, \tau) > 0$, we conclude that $\frac{\partial[\pi_i^B(\sigma_{i-1}^+) - C(x(i, \sigma_{i-1}^+))]}{\partial \sigma} > \frac{\partial[\pi_i^B(\sigma_{i-1}^-) - C(x(i, \sigma_{i-1}^-))]}{\partial \sigma}$.

Moreover, given any vector of buyers' quality functions $(\sigma(s_1, \cdot), \dots, \sigma(s_S, \cdot))$, it is possible to construct an ordered vector of sellers' qualities (τ_1, \dots, τ_T) such that there does not exist any inefficient equilibrium of the buyers' investment game.

We present the formal proof of this proposition in the Appendix. We describe here the intuition of why such result holds. The continuity of each buyer's payoff implies that, when two buyers have similar innate abilities, exactly as it is not optimal for each buyer to deviate when he is matched efficiently it is also not optimal for him to deviate when he is inefficiently assigned to a match. Indeed, the difference in buyers' qualities is almost entirely determined by the difference in the qualities of the sellers they are matched with rather than by the difference in buyers' innate ability. This implies that when the buyer of low ability has undertaken the high investment, at the purpose of being matched with the better seller, it is not worth any more for the buyer of immediately higher ability to try to outbid him. The willingness to pay of the lower ability buyer for the match with the better seller is in fact enhanced by this higher investment. Therefore the gains from outbidding this buyer are not enough to justify the high investment of the higher ability buyer. Indeed, in the Bertrand competition game each buyer is able to capture just the difference between the match surplus and the willingness to pay for the match of the runner-up buyer that in this outbidding attempt would be the low ability buyer that undertook the high investment.

Conversely, if sellers' qualities are similar then the difference in buyers qualities is almost entirely determined by the difference in buyers' innate abilities implying that it is not possible to construct an inefficient equilibrium of the buyers' investment game. The reason being that the improvement in the buyer's incentives to invest due to a match with a better seller are more than compensated by the decrease in the buyer's incentives induced by a lower innate ability of the buyer. Hence it is not optimal for two buyers of decreasing innate abilities to generate increasing qualities so as to be matched with increasing quality sellers.

We then conclude that when buyers are undertaking ex-ante match specific investments and then Bertrand compete for a match with a seller, investments are constrained efficient. If buyers are similar in innate ability inefficiencies may arise that

take the form of additional equilibria characterized by inefficient matches. However, the higher is the degree of specificity due to the buyers' characteristics with respect to the specificity due to the sellers' characteristics the less likely is this inefficiency.

We conclude this section by discussing the general case in which sellers choose their most preferred bid in the (not necessarily decreasing) order of any vector of qualities (τ_1, \dots, τ_T) .¹⁵ In this case we can prove the following corollary.

Corollary 1. *Propositions 4 and 5 hold in the general case in which sellers choose their most preferred bid in the order of any vector of sellers qualities (τ_1, \dots, τ_T) .*

The proof is presented in the Appendix and follows from the observation that neither Proposition 4 nor Proposition 5 depend on how intensely buyers compete for sellers.

6. Sellers' investments

We move now to the model in which the qualities of buyers are exogenously given by the following ordered vector $(\sigma_{(1)}, \dots, \sigma_{(S)})$, where $\sigma_{(s)} = \sigma(s)$, while the qualities of sellers are a function of sellers' ex-ante match specific investments y and the seller's identity t : $\tau(t, y)$. In this model we show that sellers' investments are not constrained efficient. Sellers under-invest since their marginal incentives to undertake investments are determined by their outside option that depends on the surplus of the match between the seller and the immediate competitor (the runner-up) to the buyer the seller is matched with in equilibrium (this match yields a strictly lower surplus than the equilibrium one).¹⁶ However, a central result is that we are able to show that equilibrium matches are such that the order of sellers' innate abilities coincides with the order of their derived qualities. In other words, all coordination problems are solved.

¹⁵Recall that seller τ_1 chooses her most preferred bid first, followed by seller τ_2 and so on till seller τ_T chooses her most preferred bid.

¹⁶We determine the size of this inefficiency in Section 7 below.

All these results crucially depend on the amount of competition in the market. Therefore in this section we almost exclusively focus on the case in which sellers select their preferred bid in the decreasing order of their innate ability. In the next section, a model that endogenously derive this ordering is used to justify this assumption.

Notice that in characterizing the equilibrium of the sellers' investment game we cannot bluntly apply Proposition 2 as the characterization of the equilibrium of the Bertrand competition subgame. Indeed, the order in which sellers choose among bids in this subgame is determined by the sellers' innate abilities rather than by their qualities. This implies that unless sellers' qualities (which are endogenously determined) have the same order of sellers' innate abilities it is possible that sellers do not choose among bids in the decreasing order of their marginal contribution to a match (at least off the equilibrium path).

For a given level of buyer's investment x_s , denote $y(t, s)$ seller t ' efficient investment when matched with buyer s defined as:

$$y(t, s) = \underset{y}{\operatorname{argmax}} v(\sigma(s), \tau(t, y)) - C(y) \quad (28)$$

We can now state the following property of the sellers' investment game.

Proposition 6. *If sellers select their most preferred bid in the decreasing order of their innate abilities the unique equilibrium of the sellers' investment game is such that seller t chooses investment $y(t, t + 1)$, as defined in (28).*

The formal proof is presented in the Appendix. However, we discuss here the intuition behind this result.

The nature of the Bertrand competition game is such that each seller is not able to capture all the match surplus but only the outside option that is determined by the willingness to pay of the runner-up buyer for the match. Since the match between a seller and her runner-up buyer yields a match surplus that is strictly lower than the equilibrium surplus produced by the same seller the share of the surplus the seller is

able to capture does not coincide with the entire surplus of the match. This implies that sellers under-invest rendering the equilibrium investment choice inefficient.

Corollary 2. *When sellers undertake ex-ante investments and choose their most preferred bid in the decreasing order of their innate abilities then each seller $t = 1, \dots, T$ chooses an inefficient investment level $y(t, t+1)$. The investment $y(t, t+1)$ is strictly lower than the investment $y(t, t)$ that would be efficient for seller t to choose given the equilibrium match of buyer t with seller t .*

Proof: The result follows from Proposition 6, the definition of efficient investment (28) when buyer t matches with seller t , and condition (A.38) in the Appendix. ■

In contrast with the case in which buyers undertake ex-ante investments, in this framework the equilibrium of the Bertrand competition game is unique and characterized by the absence of any coordination failure.

Corollary 3. *When sellers undertake ex-ante investments the unique equilibrium of the Bertrand competition game is characterized by matches between buyer t and seller t , $t = 1, \dots, T$ that are assortative both in terms of sellers' innate abilities and of sellers' qualities.*

Proof: The result follows immediately from Proposition 6 above. ■

Two features of the model may explain why equilibria that exhibit coordination failures of the type described in Section 5 for the case of buyers' investments do not exist. First, as argued above, each seller's payoff is completely determined by the seller's outside option and hence independent of the identity and quality of the buyer the seller is matched with. Second, sellers choose their bid in the decreasing order of their innate abilities hence this order is independent of sellers' investments. These two features of the model together with positive assortative matching (Lemma 1 above) imply that when a seller chooses an investment that yields a quality higher than the one of the seller with a lower identity (higher innate ability) it modifies the set of

unmatched buyers, and hence of bids among which the seller chooses, only of the bid of the buyer the seller will be matched with in equilibrium. Hence this change will not affect the outside option and therefore the payoff of this seller implying that the optimal investment cannot exceed the optimal investment of the seller with higher innate ability. Therefore an equilibrium with inefficient investment does not exist.

An interesting issue is whether this uniqueness is preserved if we modify the extensive form of the Bertrand competition game and in particular the order in which sellers choose their most preferred bid.

Notice first that the intuition we just described does not hold if sellers choose their bid in the decreasing order of their qualities and not of their innate abilities. In this case the order in which sellers choose their most preferred bid is endogenously determined. An argument similar to the one used in the analysis of the buyers' investment game (Proposition 4 above) will then show that equilibrium with efficient matches always exist. However there may exist multiple equilibria that exhibit inefficient matches.

Consider now the general case in which sellers choose their bid in the order of the permutation (t_1, \dots, t_T) . For simplicity we focus on the case in which sellers choose their bids in the increasing order of their innate ability: $t_1 = T, \dots, t_T = 1$.¹⁷ Notice first that an efficient equilibrium exists in which sellers qualities have the same order of sellers' innate abilities. Consider such an equilibrium of the sellers' investment game. As argued in Section 4 above, in this case the runner-up buyer to every seller is the highest quality buyer that does not match with any seller in equilibrium. This implies that each seller t 's payoff is

$$v(\sigma_{(T+1)}, \tau(t, y)). \quad (29)$$

Therefore each seller's net payoff function $v(\sigma_{(T+1)}, \tau(t, y)) - C(y)$ has a unique maximum at $y(t, T + 1)$. This implies that sellers' equilibrium investments and hence

¹⁷Using Propositions 1 and 3 above this analysis can be generalized to the case in which sellers choose their most preferred bid in the order of any permutation (t_1, \dots, t_T) .

sellers' qualities have the same order of sellers' innate abilities.

Notice however that inefficient equilibria are not ruled out. The logic behind these equilibria can be described as follows. Consider seller t and assume that this seller chooses a level of investment yielding a quality higher than the one chosen by seller $k < t$. Notice now that, from Lemma 1, in the case in question this change in investment affects the equilibrium matches of all the buyers with identities between t and k that are un-matched when it is seller t 's turn to choose a bid. This implies that the outside option of seller t will also be affected by this increase in investment creating the conditions for an equilibrium characterized by inefficient matches.

7. The Inefficiency of Sellers' Investments

7.1. *The Intensity of Competition*

The analysis above shows that if, in the Bertrand competition game, sellers select their most preferred bid in the decreasing order of their innate ability, as opposed to any other order, competition among buyers for matches is at its peak.

The question we ask in this section is whether we expect such a high intensity of competition to arise when we endogenize the order in which sellers select their most preferred bid. We do this by allowing sellers to compete for the order in which they select their bid at an ex-ante stage of the game that precedes the sellers' investment decision. We are able to show that there is an equilibrium of this ex-ante sellers' competition game in which the equilibrium order is the decreasing order of the sellers' innate abilities. Moreover, if sellers innate abilities are sufficiently far apart, this is the unique equilibrium of the ex-ante sellers' competition game.

As discussed at the end of Section 6 above, when sellers select their most preferred bid in any other order but the decreasing order of their innate abilities, multiple equilibria may arise in the sellers' investment game. These equilibria are characterized by inefficient matches. In our analysis below, whatever the order in which sellers select their most preferred bid, we restrict attention to the equilibrium of the sellers' investment game with no coordination failures. Adapting the arguments presented in

the proof of Propositions 4 and 5 it is possible to show, as in Section 6 above, that this equilibrium always exists. This equilibrium is also unique if the sellers' innate abilities are sufficiently far apart. In other words, we restrict ourselves to the case in which sellers' innate abilities are sufficiently far apart so that the unique equilibrium of the sellers' investment game is such that the order of sellers' innate abilities coincides with the order of their induced qualities whatever the order in which sellers select their most preferred bid. The equilibrium of the ex-ante sellers' competition game is then summarized by the order of the vector of sellers' qualities (τ_1, \dots, τ_T) where the seller of quality τ_1 is the first seller to select her most preferred bid while τ_T is the last seller to select her most preferred bid.

Consider now an equilibrium of this ex-ante sellers' competition game (τ_1, \dots, τ_T) that differs from the decreasing order of the sellers' innate abilities or the decreasing order of their qualities (given the equilibrium of the sellers' investment game). Then we can find a t^* such that

$$\tau_{t^*} < \tau_{t^*+1} > \tau_{t^*+2} > \dots > \tau_T \quad (30)$$

where $t^* \in \{1, \dots, T-1\}$. Now, it is clear that there exists a seller of quality $\tau_{t'}$ such that $t' \in \{t^*+1, \dots, T\}$ and

$$\tau_{t'} > \tau_{t^*} > \tau_{t'+1}$$

We want to compare the equilibrium payoffs to sellers t^* and t' if the equilibrium order of the ex-ante sellers' competition game is the one described in (30) above with their payoffs if the two sellers t^* and t' swap position in the order in which they choose their most preferred bid. Notice that in the latter case all sellers can be expected to choose a different investment level and hence will be associated with a different quality level. We denote the quality levels chosen when sellers select their most preferred bid according to this new order $\tilde{\tau}_t$.

Notice first that from Lemma 1, given that we restrict attention to equilibria of the sellers' investment game with efficient matches, if sellers follow this new order in selecting their bid each seller will be matched in equilibrium with the same buyer.

Secondly, Lemma 2 and Proposition 1 (in particular the sellers's payoffs in (11) above) imply that sellers $\{\tilde{\tau}_{t'+1}, \dots, \tilde{\tau}_T\}$ will have the same runner-up buyer, get the same payoff and choose the same investment as in the order described in (30) above: $\tau_t = \tilde{\tau}_t$ for every $t \in \{t' + 1, \dots, T\}$. Further, Lemma 2 implies that the seller of quality $\tilde{\tau}_{t^*}$ has the same runner-up buyer and, from (11) above, the same payoff as the seller of quality τ_{t^*} . In other words,

$$\tau_{t^*} = \tilde{\tau}_{t^*}$$

and

$$\pi_{t^*}^S = \tilde{\pi}_{t^*}^S$$

where we denote $\pi_{t^*}^S$ the equilibrium payoff to the seller of quality τ_{t^*} and $\tilde{\pi}_{t^*}^S$ the equilibrium payoff of the seller of quality $\tilde{\tau}_{t^*}$. Thus, seller t^* obtains the same payoff if the equilibrium order of the ex-ante sellers' competition game is the one described in (30) above or the one in which seller t^* and t' swap position.

Consider now seller t' . The payoff function in (11) is identical in the two potential orders with the sole exception that, from Lemma 2 above, when t^* and t' swap positions, the runner-up buyer to seller t' is the buyer that in equilibrium matches with seller t^* . Therefore from (11) above we obtain:

$$\tilde{\pi}_{t'}^S - \pi_{t'}^S = v(\sigma_{t^*}, \tilde{\tau}_{t'}) - v(\sigma_{t^*}, \tilde{\tau}_{t^*}) + v(\sigma_{t'+1}, \tilde{\tau}_{t^*}) - v(\sigma_{t'+1}, \tau_{t'}) \quad (31)$$

Notice that, as $\tau_{t'} > \tilde{\tau}_{t^*}$ and $\sigma_{t^*} > \sigma_{t'+1}$ using the complementarity assumption, $v_{12}(\sigma, \tau) > 0$, we can conclude that if $\tau_{t'} = \tilde{\tau}_{t'}$ then the difference in (31) is strictly positive. But since the runner up buyer of the sellers of quality $\tilde{\tau}_{t'}$ is of higher quality σ_{t^*} than the runner-up buyer of quality $\sigma_{t'+1}$ of the seller of quality $\tau_{t'}$ when sellers t^* and t' swap their positions, seller t' will choose a higher investment: $\tilde{\tau}_{t'} > \tau_{t'}$. Hence, the difference in (31) is further magnified and strictly positive. In other words, seller t' strictly gains from swapping position in the order in which sellers select their most preferred bid with seller t^* .

In essence, seller t^* 's runner-up buyer has not changed as a consequence of the swap and hence seller t^* does not lose out from the change. At the same time, by

swapping position, seller t' improves the quality of her runner-up buyer, increasing in this way her payoff, and further gains by being able to exploit this better potential match at the investment stage.

We can now summarize our findings in the following proposition.

Proposition 7. *If sellers select their most preferred bid in any order other than the decreasing order of their innate abilities then there always exists a pair of sellers who gain, one weakly and one strictly, by swapping their position in the order in which they select their bid.*

Notice that Proposition 7 above implies that there cannot exist an equilibrium order of the ex-ante sellers' competition game that differs from the decreasing order of the sellers' innate abilities. Indeed, a minimal requirement for any model of sellers' competition for a position is that there does not exist a pair of competitors that strictly gains by swapping position.¹⁸

Therefore, when the intensity of buyers' competition for sellers is endogenized, the unique equilibrium is such that competition will be at its peak. This result provides us with a justification for restricting attention, in the remainder of this section, to the properties of the equilibrium of the model in which sellers select their bid in the decreasing order of their innate abilities.

7.2. *The Size of the Inefficiency*

In this section we evaluate the size of the inefficiency generated by sellers' under-investment and characterized in Section 6 above. In particular we argue that when competition among buyers for sellers is at its peak this inefficiency is small in a well defined sense. In particular, we show that when sellers choose their most preferred bid in the decreasing order of their innate abilities the overall inefficiency generated by sellers' equilibrium under-investment is strictly lower than the inefficiency induced

¹⁸Different extensive forms of the sellers' competition game are consistent with Proposition 7 above. For example, sellers' competition for a position could be structured as a Bertrand competition game in the style of the buyers' competition game analyzed in Section 4 above.

by the under-investment of one seller (the best one) if it matches in isolation with the best buyer.

To be able to proceed we need stronger assumptions on the responsiveness of sellers' investments to both the buyers' and sellers' identities and on each match surplus function.

The first assumption, labelled *responsive complementarity*, can be described as follows. Recall that seller t 's efficient investment $y(t, s)$ when matches with buyer s is defined in (28) above. In other words $y(t, s)$ satisfies:

$$v_2(\sigma(s), \tau(t, y(t, s))) \tau_2(t, y(t, s)) = C''(y(t, s)) \quad (32)$$

Seller t 's investment $y(t, s)$ satisfies responsive complementarity if and only if:

$$\frac{\partial}{\partial t} \left(\frac{\partial y(t, s)}{\partial s} \right) > 0. \quad (33)$$

In other words:

$$\frac{\partial}{\partial t} \left(-\frac{v_{12} \sigma_1 \tau_2}{v_{22}(\tau_2)^2 + v_2 \tau_{22} - C''} \right) > 0 \quad (34)$$

where the first and second order derivatives τ_2 and τ_{22} are computed at $(t, y(t, s))$, the derivatives v_h and v_{hk} , $h, k \in \{1, 2\}$ are computed at $(\sigma(s), \tau(t, y_t(s)))$ and C'' is computed at $y(t, s)$.

We label the second assumption *marginal complementarity*. This assumption requires that the marginal surplus generated by a higher seller's quality satisfies:

$$\frac{\partial^2 v_2(\sigma, \tau)}{\partial \sigma \partial \tau} > 0. \quad (35)$$

or $v_{122} > 0$. Notice that both responsive and marginal complementarity are satisfied by a standard iso-elastic surplus function $v(\cdot, \cdot)$.

Denote $\omega(s, t)$ the net surplus function when buyer t matches with seller t and the

seller's investment is $y(t, s)$ as in (28) above.

$$\omega(s, t) = v(\sigma(t), \tau(t, y(t, s))) - C(y(s, t)). \quad (36)$$

Clearly, in definition (36), the investment $y(t, s)$ maximizes the net surplus of a match (between buyer s and seller t) that might differ from the match with buyer t in which seller t is involved.

From Corollary 2 above we know that each seller will under-invest and choose an investment $y(t, t + 1) < y(t, t)$. Hence the inefficiency associated with each seller t 's investment decision is characterized by the difference between the match surplus generated by the efficient investment $y(t, t)$ and the match surplus generated by the equilibrium investment $y(t, t + 1)$:

$$\omega(t, t) - \omega(t, t + 1).$$

Therefore the inefficiency of the equilibrium investments by all sellers is given by

$$L = \sum_{t=1}^T \omega(t, t) - \sum_{t=1}^T \omega(t, t + 1). \quad (37)$$

How large is this loss L ? First, notice that the difference between the efficient investment $y(t, t)$ and the equilibrium investment $y(t, t + 1)$ is approximately proportional to the difference in characteristics between buyer t and $t + 1$ (given that $y(t, s)$ as defined in (28) is differentiable in s). On the other hand, as $y(t, t)$ solves (32), the difference between the efficient surplus $\omega(t, t)$ and the equilibrium surplus $\omega(t, t + 1)$ will be approximately proportional to the *square* of the difference between $y(t, t)$ and $y(t, t + 1)$ which will be small if buyer t and buyer $t + 1$ have similar characteristics. To give an example of how this affects L , consider a situation where the characteristics of a buyer are captured by a real number c with buyers 1 through S having characteristics which are evenly spaced between \bar{c} and \underline{c} . How is L affected by the size of the market T ? The difference between $y(t, t)$ and $y(t, t + 1)$ is approximately proportional to $[(\bar{c} - \underline{c})/T]$ and the difference between $\omega(t, t)$ and $\omega(t, t + 1)$ will be

approximately proportional to $[(\bar{c} - \underline{c})/T]^2$. Summing over t then gives a total loss L that is proportional to $[(\bar{c} - \underline{c})^2/T]$: in large markets the aggregate inefficiency created by sellers' investments will be arbitrarily small.¹⁹

This is a result that changes the degree of specificity of the sellers' investment choices. Increasing the number and hence the density of buyers evenly spaced in the interval $[\underline{c}, \bar{c}]$ is equivalent to introducing buyers with closer and closer characteristics. This is equivalent to reducing the loss in productivity generated by the match of a seller that choose an investment so as to be matched with the buyer that is immediately below in characteristics levels. Hence, there is a sense in which this result is not fully satisfactory since we know that if each seller's investment is general in nature the investment choices are efficient.

Therefore, in the rest of this section, we identify an upper-bound on the aggregate inefficiency present in the economy that is independent of the number of sellers and does not alter the specificity of the buyers investment choices. Whatever the size of T , it is possible to get a precise upper-bound on the loss L . Indeed, the inefficiency created by the sellers' equilibrium under-investment is less than that which could be created by the under-investment of only one seller (the best seller 1) in a match with only one buyer (the best one labelled 1).

Proposition 8. *Assume that there are at least two sellers ($T \geq 2$). Let M be the efficiency loss resulting from seller 1 choosing an investment level given by $y(1, T+1)$, as defined in (28):*

$$M = \omega(1, 1) - \omega(1, T+1). \quad (38)$$

If both responsive complementarity, as in (33), and marginal complementarity, as in (35), are satisfied then

$$L < M. \quad (39)$$

The formal proof is presented in the Appendix, while the intuition of Proposition 8 can be described as follows. As a result of the Bertrand competition game sellers

¹⁹See Kamecke (1992).

have incentive to invest in match specific investments with the purpose of improving their outside option: the maximum willingness to pay of the runner-up buyer to the seller. This implies that the under-investment of each seller is relatively small. The total inefficiency is then obtained by aggregating these relatively small under-investments. Given the decreasing returns to investment and the assumptions on how optimal sellers' investments change across different matches, the sum of the loss in surplus generated by these almost optimal investments is clearly dominated by the loss in surplus generated by the unique under-investment of the best seller matched with the best buyer. Indeed, the seller's investment choice in the latter case is very far from the optimal level (returns from a marginal increase of investment are very high).

8. Concluding Remarks

When buyers and sellers can undertake match specific investments, Bertrand competition for matches may help solve the hold-up and coordination problems generated by the absence of fully contingent contracts. In this paper, we have uncovered a number of characterization results that highlight how competition may solve, or at least attenuate, the impact of these problems.

When buyers choose investments that precede Bertrand competition then the buyers' investment choices are constrained efficient. However, coordination failure inefficiencies may arise that take the form of multiple equilibria and only one of these equilibria is characterized by efficient matches: there may exist inefficient equilibria that exhibit matches such that buyers with lower innate ability invest more than better buyers at the sole purpose of being matched with a higher quality seller.

If instead sellers choose investments that precede the Bertrand competition game a different set of inefficiencies may arise. When buyers are competed for in decreasing order of innate ability then the equilibrium of the Bertrand competition game is unique and involves no coordination failures. However, sellers choose an inefficient level of investment given the equilibrium match they are involved in. In this case, we show first that the decreasing order of sellers' innate abilities is the one that maximizes

buyers' competition for sellers and is the order that arises in equilibrium if sellers at an earlier stage compete for the position in the order in which they select their most preferred bid. We then show that the aggregate inefficiency due to sellers' under-investments is low in the sense that is bounded above by the inefficiency that would be induced by the sole under-investment of the best seller matched with the best buyer. In other words the inefficiencies due to the hold-up problem do not cumulate in the presence of buyers' competition for matches.

Consider now what will happen in this environment if both buyers and sellers undertake ex-ante investments. Buyers' investments will still be constrained efficient while sellers' investments, although inefficient, can still be near efficient (when competition is in the decreasing order of buyers' innate ability and the appropriate equilibrium is selected). However, if both buyers and sellers undertake ex-ante investments then the inefficiency that takes the form of multiple equilibria, some of them characterized by inefficient matches, can still arise.

We conclude with the observation that the extensive form of the Bertrand competition game we use in the paper coincides with a situation in which sellers are sequentially auctioned off to buyers. Our result can then be re-interpreted as applying to a model of perfect information sequential auctions in which buyers' valuations for each seller and the value of each auctioned-off seller can be enhanced by ex-ante investments.

Appendix

Proof of Lemma 2: We concentrate on the case where all sellers and all buyers have different induced quality. Bids are made as part of a trembling-hand-perfect equilibrium. We will prove the result by induction on the number of sellers still to be matched. Without any loss in generality, we take $S = T + 1$. Consider the (last) stage T of the Bertrand competition game. In this stage only two buyers are unmatched and from Lemma 1 have qualities σ_T and σ_{T+1} . Clearly in this case the only possible runner-up to seller T is the buyer of quality σ_{T+1} , and given that by Lemma 1 $\sigma_T > \sigma_{T+1}$ the quality of this buyer satisfies (5) above.

Further this stage of the Bertrand competition game is a simple decision problem for seller T that has to choose between the bids submitted by the two buyers with qualities σ_T and σ_{T+1} . Let $b(\sigma_T)$, respectively $b(\sigma_{T+1})$, be their bids. Seller T clearly chooses the highest of these two bids.

Buyer of quality σ_{T+1} generates surplus $v(\sigma_{T+1}, \tau_T)$ if selected by seller T while the buyer of quality σ_T generates surplus $v(\sigma_T, \tau_T)$ if selected. This implies that $v(\sigma_{T+1}, \tau_T)$ is the maximum willingness to bid of the runner-up buyer σ_{T+1} , while $v(\sigma_T, \tau_T)$ is the maximum willingness to bid of the buyer of quality σ_T . Notice that from $\sigma_T > \sigma_{T+1}$ and $v_1(\sigma, \tau) > 0$ we have:

$$v(\sigma_T, \tau_T) > v(\sigma_{T+1}, \tau_T).$$

Buyer σ_T therefore submits a bid equal to the minimum necessary to outbid buyer σ_{T+1} . In other words the equilibrium bid of buyer σ_T coincides with the equilibrium bid of buyer σ_{T+1} : $b(\sigma_T) = b(\sigma_{T+1})$. Buyer σ_{T+1} , on his part, has an incentive to deviate and outbid buyer σ_T for any bid $b(\sigma_T) < v(\sigma_{T+1}, \tau_T)$. Therefore the unique equilibrium is such that both buyers' equilibrium bids are:²⁰

$$b(\sigma_T) = b(\sigma_{T+1}) = v(\sigma_{T+1}, \tau_T)$$

Consider now the stage $t < T$ of the Bertrand competition game. The induction hypothesis is that the runner-up buyer for every seller of quality $\tau_{t+1}, \dots, \tau_T$ is defined in (5) above. Further, the shares of surplus accruing to the sellers of qualities τ_j , $j = t + 1, \dots, T$ and to the buyers of qualities σ_j , $j = t + 1, \dots, S$ are:

$$\hat{\pi}_{\sigma_j}^B = [v(\sigma_j, \tau_j) - v(\sigma_{r(j)}, \tau_j)] + \hat{\pi}_{\sigma_{r(j)}}^B \quad (\text{A.1})$$

$$\hat{\pi}_{\tau_j}^S = v(\sigma_{r(j)}, \tau_j) - \hat{\pi}_{\sigma_{r(j)}}^B. \quad (\text{A.2})$$

²⁰This is just one of a whole continuum of subgame perfect equilibria of this simple Bertrand game *but* the unique trembling-hand-perfect equilibrium. Trembling-hand-perfection is here used in a completely standard way to insure that buyer σ_{T+1} does not choose an equilibrium bid (not selected by seller T) in excess of his maximum willingness to pay.

From Lemma 1 the buyer of quality σ_t will match with the seller of quality τ_t which implies that the runner-up buyer for seller τ_t has to be one of the buyers with qualities $\sigma_{t+1}, \dots, \sigma_{T+1}$. Each buyer will bid an amount for every seller which gives him the same payoff as he receives in equilibrium. To prove that the quality of the runner-up buyer satisfies (5) we need to rule out that the quality of the runner-up buyer is $\sigma_{r(t)} > \sigma_t$ and, if $\sigma_{r(t)} \leq \sigma_t$, that there exist an other buyer of quality $\sigma_i \leq \sigma_t$ such that $i > t$ and $\sigma_i > \sigma_{r(t)}$.

Assume first by way of contradiction that $\sigma_{r(t)} > \sigma_t$. Then the willingness to pay of the runner-up buyer for the match with seller τ_t is the difference between the surplus generated by the match of the runner-up buyer of quality $\sigma_{r(t)}$ and the seller of quality τ_t minus the payoff that the buyer would get according to the induction hypothesis by moving to stage $r(t)$ of the Bertrand competition game:

$$v(\sigma_{r(t)}, \tau_t) - \hat{\pi}_{\sigma_{r(t)}}^B. \quad (\text{A.3})$$

From the induction hypothesis, (A.1), we get that the payoff $\hat{\pi}_{\sigma_{r(t)}}^B$ is:

$$\hat{\pi}_{\sigma_{r(t)}}^B = v(\sigma_{r(t)}, \tau_{r(t)}) - v(\sigma_{r^2(t)}, \tau_{r(t)}) + \hat{\pi}_{\sigma_{r^2(t)}}^B \quad (\text{A.4})$$

where, from the induction hypothesis, $\sigma_{r^2(t)} < \sigma_{r(t)}$. Substituting (A.4) into (A.3) we get that the willingness to pay of a runner-up buyer of quality $\sigma_{r(t)}$ for the match with the seller of quality τ_t can be written as:

$$v(\sigma_{r(t)}, \tau_t) - v(\sigma_{r(t)}, \tau_{r(t)}) + v(\sigma_{r^2(t)}, \tau_{r(t)}) - \hat{\pi}_{\sigma_{r^2(t)}}^B. \quad (\text{A.5})$$

Consider now the willingness to pay of the buyer of quality $\sigma_{r^2(t)}$ for the match with the same seller of quality τ_t . This is

$$v(\sigma_{r^2(t)}, \tau_t) - \hat{\pi}_{\sigma_{r^2(t)}}^B. \quad (\text{A.6})$$

By definition of runner-up buyer the willingness to pay of the buyer of quality $\sigma_{r(t)}$, as in (A.5), must be greater or equal than the willingness to pay of the buyer of quality $\sigma_{r^2(t)}$ as in (A.6). This inequality is satisfied if and only if:

$$v(\sigma_{r(t)}, \tau_t) + v(\sigma_{r^2(t)}, \tau_{r(t)}) \geq v(\sigma_{r(t)}, \tau_{r(t)}) + v(\sigma_{r^2(t)}, \tau_t). \quad (\text{A.7})$$

Since $\sigma_{r(t)} > \sigma_t$ then from Lemma 1 $\tau_{r(t)} > \tau_t$. The latter inequality together with $\sigma_{r(t)} > \sigma_{r^2(t)}$ allow us to conclude that (A.7) is a contradiction to the complementarity assumption $v_{12}(\sigma, \tau) > 0$.

Assume now by way of contradiction that the $\sigma_{r(t)} \leq \sigma_t$ but there exists another buyer of quality $\sigma_i \leq \sigma_t$ such that $i > t$ and $\sigma_i > \sigma_{r(t)}$. The definition of runner-up buyer implies that his willingness to pay, as in (A.3), for the match with the seller of quality τ_t is greater than the willingness to pay

$v(\sigma_i, \tau_t) - \hat{\pi}_{\sigma_i}^B$ of the buyer of quality σ_i , for the same match:

$$v(\sigma_{r(t)}, \tau_t) - \hat{\pi}_{\sigma_{r(t)}}^B \geq v(\sigma_i, \tau_t) - \hat{\pi}_{\sigma_i}^B. \quad (\text{A.8})$$

Moreover, for $(\sigma_{r(t)}, \tau_{r(t)})$ to be an equilibrium match buyer $\sigma_{r(t)}$ should have no incentive to be matched with seller τ_i instead. This implies, using an argument identical to the one presented in the proof of Lemma 1, that the following necessary condition needs to be satisfied:

$$\hat{\pi}_{\sigma_{r(t)}}^B = v(\sigma_{r(t)}, \tau_{r(t)}) - b(\tau_{r(t)}) \geq v(\sigma_{r(t)}, \tau_i) - b(\tau_i); \quad (\text{A.9})$$

where $b(\tau_{r(t)})$ and $b(\tau_i)$ are the equilibrium bids accepted by seller $\tau_{r(t)}$, respectively τ_i . Further, the equilibrium payoff to buyer σ_i is:

$$\hat{\pi}_{\sigma_i}^B = v(\sigma_i, \tau_i) - b(\tau_i). \quad (\text{A.10})$$

Substituting (A.9) and (A.10) into (A.8) we obtain that for (A.8) to hold the following necessary condition needs to be satisfied:

$$v(\sigma_{r(t)}, \tau_t) + v(\sigma_i, \tau_i) \geq v(\sigma_i, \tau_t) + v(\sigma_{r(t)}, \tau_i). \quad (\text{A.11})$$

Since by assumption $\sigma_t \geq \sigma_i$ from Lemma 1 $\tau_t > \tau_i$. The latter inequality together with $\sigma_i > \sigma_{r(t)}$ imply that (A.11) is a contradiction to the complementarity assumption $v_{12}(\sigma, \tau) > 0$. This concludes the proof that the quality of the runner-up buyer for seller τ_t satisfies (5).

An argument similar to the one used in the analysis of stage T of the Bertrand competition subgame concludes the proof of Lemma 2 by showing that the buyer of quality σ_t submits in equilibrium a bid equal to the willingness to pay of the runner-up buyer to seller τ_t as in (A.3). This bid is the equilibrium payoff to the seller of quality τ_t and coincides with (7). The equilibrium payoff to the buyer of quality σ_t is then the difference between the match surplus $v(\sigma_t, \tau_t)$ and the equilibrium bid in (A.3) as in (6). ■

Lemma A.1. *Given any ordered vector of sellers' qualities (τ_1, \dots, τ_T) and the corresponding vector of buyers' qualities $(\sigma_1, \dots, \sigma_S)$ we have that for every $t = 1, \dots, T - 1$ and every $m = 1, \dots, T - t$:*

$$v(\sigma_{t+1}, \tau_t) - \sum_{h=1}^m [v(\sigma_{t+h}, \tau_{t+h}) - v(\sigma_{t+h+1}, \tau_{t+h})] > v(\sigma_{t+m}, \tau_t) \quad (\text{A.12})$$

Proof: We prove this result by induction. In the case $m = 1$ inequality (A.12) becomes:

$$v(\sigma_{t+1}, \tau_t) - v(\sigma_{t+1}, \tau_{t+1}) + v(\sigma_{t+2}, \tau_{t+1}) > v(\sigma_{t+2}, \tau_t)$$

which is satisfied by the complementarity assumption $v_{12}(\sigma, \tau) > 0$, given that $\sigma_{t+1} > \sigma_{t+2}$ and $\tau_t > \tau_{t+1}$. Assume now that for every $1 \leq n < m$ the following condition holds:

$$v(\sigma_{t+1}, \tau_t) - \sum_{h=1}^n [v(\sigma_{t+h}, \tau_{t+h}) - v(\sigma_{t+h+1}, \tau_{t+h})] > v(\sigma_{t+n}, \tau_t) \quad (\text{A.13})$$

We need to show that (A.12) holds for $m = n + 1$. Inequality (A.12) can be written as:

$$\begin{aligned} v(\sigma_{t+1}, \tau_t) &- \sum_{h=1}^n [v(\sigma_{t+h}, \tau_{t+h}) - v(\sigma_{t+h+1}, \tau_{t+h})] - \\ &- [v(\sigma_{t+n+1}, \tau_{t+n+1}) - v(\sigma_{t+n+2}, \tau_{t+n+1})] > v(\sigma_{t+n+1}, \tau_t) \end{aligned} \quad (\text{A.14})$$

Substituting the induction hypothesis (A.13) into (A.14) we obtain:

$$\begin{aligned} v(\sigma_{t+1}, \tau_t) &- \sum_{h=1}^n [v(\sigma_{t+h}, \tau_{t+h}) - v(\sigma_{t+h+1}, \tau_{t+h})] - \\ &- [v(\sigma_{t+n+1}, \tau_{t+n+1}) - v(\sigma_{t+n+2}, \tau_{t+n+1})] > \\ &> v(\sigma_{t+n+1}, \tau_t) - v(\sigma_{t+n+1}, \tau_{t+n+1}) + v(\sigma_{t+n+2}, \tau_{t+n+1}) \end{aligned} \quad (\text{A.15})$$

Notice now that the complementarity assumption $v_{12}(\sigma, \tau) > 0$ and the inequalities $\sigma_{t+n+1} > \sigma_{t+n+2}$, $\tau_t > \tau_{t+n+1}$ imply:

$$v(\sigma_{t+n+1}, \tau_t) - v(\sigma_{t+n+1}, \tau_{t+n+1}) + v(\sigma_{t+n+2}, \tau_{t+n+1}) > v(\sigma_{t+n+2}, \tau_t) \quad (\text{A.16})$$

Substituting (A.16) into (A.15) we conclude that (A.12) holds for $m = n + 1$. ■

Proof of Proposition 3: Consider the vectors of subsequent runner-up buyers $(\sigma_t, \dots, \sigma_{T+1})$ and $(\sigma'_t, \sigma'_{r(t)}, \dots, \sigma'_{r^{\rho'_t}(t)})$. From Lemma 1 and the assumption $\tau'_t = \tau_t$ we get that $\sigma_t = \sigma'_t$. Moreover from (5) we have that $\sigma_{T+1} = \sigma'_{r^{\rho'_t}(t)}$ and there exist an index $\ell(r^k(t)) \in \{t+1, \dots, T+1\}$ such that

$$\sigma_{\ell(r^k(t))} = \sigma'_{r^k(t)}$$

for every $k = 0, \dots, \rho'_t$, where $r^0(t) = t$. In other words, the characterization of the runner-up buyer (5) implies that the elements of the vector $(\sigma'_t, \sigma'_{r(t)}, \dots, \sigma'_{r^{\rho'_t}(t)})$ are a subset of the elements of the vector $(\sigma_t, \sigma_{t+1}, \dots, \sigma_{T+1})$. Lemma 1 then implies that

$$\tau_{\ell(r^k(t))} = \tau'_{r^k(t)}$$

for every $k = 0, \dots, \rho'_t$. Therefore we can rewrite the payoff to seller τ'_t , as in (11), in the following

way:

$$v(\sigma_{\ell(r(t))}, \tau_{\ell(t)}) - \sum_{k=1}^{\rho'_t} [v(\sigma_{\ell(r^k(t))}, \tau_{\ell(r^k(t))}) - v(\sigma_{\ell(r^{k+1}(t))}, \tau_{\ell(r^k(t))})] . \quad (\text{A.17})$$

Define now δ_k be an integer number such that $\ell(r^k(t)) + \delta_k = \ell(r^{k+1}(t))$. Then Lemma A.1 implies that:

$$\begin{aligned} v(\sigma_{\ell(r^k(t))+1}, \tau_{\ell(r^k(t))}) - \sum_{h=1}^{\delta_k-1} [v(\sigma_{\ell(r^k(t))+h}, \tau_{\ell(r^k(t))+h}) - v(\sigma_{\ell(r^k(t))+h+1}, \tau_{\ell(r^k(t))+h})] &> \\ > v(\sigma_{\ell(r^{k+1}(t))}, \tau_{\ell(r^k(t))}) \end{aligned} \quad (\text{A.18})$$

for every $k = 0, \dots, \rho'_t - 1$. Substituting (A.18) into (A.17) we obtain (18). ■

Proof of Proposition 4: We prove this result in three steps. We first show that the buyers' equilibrium qualities $\sigma(i, x_i(i))$ associated with the equilibrium $s_i = i$ satisfy condition (27). We then show that the net payoff to buyer i associated with any given quality σ of this buyer is continuous in σ . This result is not obvious since, from Lemma 1 — given the investment choices of other buyers — buyer i can change his equilibrium match by changing his quality σ . Finally, we show that this net payoff has a unique global maximum and this maximum is such that the corresponding quality σ is in the interval in which buyer i is matched with seller i . These steps clearly imply that each buyer i has no incentive to deviate and choose an investment different from the one that maximizes his net payoff and yields an equilibrium match with seller i .

Let $\pi_i^B(\sigma) - C(x(i, \sigma))$ be the net payoff to buyer i where $x(i, \sigma)$ denotes buyer i 's investment level associated with quality σ :

$$\sigma(i, x(i, \sigma)) \equiv \sigma. \quad (\text{A.19})$$

Step 1. Buyer i 's equilibrium quality $\sigma(i, x_i(i))$ is such that:

$$\sigma(i, x_i(i)) = \sigma_i < \sigma(i-1, x_{i-1}(i-1)) = \sigma_{i-1}, \quad \forall i = 2, \dots, S.$$

The proof follows directly from Lemma 4 above.

Step 2. The net payoff $\pi_i^B(\sigma) - C(x(i, \sigma))$ is continuous in σ .

Let $(\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_S)$ be the given ordered vector of the qualities of the buyers, other than i . Notice that if $\sigma \in (\sigma_{i-1}, \sigma_{i+1})$ by Lemma 1 buyer i is matched with the seller of quality τ_i . Then by Proposition 2 and the definition of $v(\cdot, \cdot)$, $C(\cdot)$, $\sigma(\cdot, \cdot)$ and (A.19) the payoff function $\pi_i^B(\sigma) - C(x(i, \sigma))$ is continuous in σ .

Consider now the limit for $\sigma \rightarrow \sigma_{i-1}^-$ from the right of the net payoff to buyer i when it is matched with the seller of quality τ_i , $\sigma \in (\sigma_{i+1}, \sigma_{i-1})$. From (12) this limit is

$$\begin{aligned} \pi_i^B(\sigma_{i-1}^-) - C(x(i, \sigma_{i-1}^-)) &= v(\sigma_{i-1}, \tau_i) - v(\sigma_{i+1}, \tau_i) + \\ &+ \sum_{h=i+1}^T [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)] - C(x(i, \sigma_{i-1})). \end{aligned} \quad (\text{A.20})$$

Conversely, if $\sigma \in (\sigma_{i-1}, \sigma_{i-2})$ then by Lemma 1 buyer i is matched with the seller of quality τ_{i-1} and the payoff is continuous in this interval. Then from (12) the limit for $\sigma \rightarrow \sigma_{i-1}^+$ from the left of the net payoff to buyer i when matched with the seller of quality τ_{i-1} is

$$\begin{aligned} \pi_i^B(\sigma_{i-1}^+) - C(x(i, \sigma_{i-1}^+)) &= v(\sigma_{i-1}, \tau_{i-1}) - v(\sigma_{i-1}, \tau_{i-1}) + \\ &+ v(\sigma_{i-1}, \tau_i) - v(\sigma_{i+1}, \tau_i) + \\ &+ \sum_{h=i+1}^T [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)] - C(x(i, \sigma_{i-1})). \end{aligned} \quad (\text{A.21})$$

In this case while the buyer of quality σ is matched with the seller of quality τ_{i-1} the buyer of quality σ_{i-1} is matched with the seller of quality τ_i .

Equation (A.20) coincides with equation (A.21) since the first two terms of the left-hand-side of equation (A.21) are identical. A similar argument shows continuity of the net payoff function at $\sigma = \sigma_h$, $h = 1, \dots, i-2, i+1, \dots, N$.

Step 3. The net surplus function $\pi_i^B(\sigma) - C(x(i, \sigma))$ has a unique global maximum in the interval $(\sigma_{i+1}, \sigma_{i-1})$.

Notice first that in the interval $(\sigma_{i+1}, \sigma_{i-1})$, by Lemma 1 and Proposition 2, the net payoff $\pi_i^B(\sigma) - C(x(i, \sigma))$ takes the following expression.

$$\begin{aligned} \pi_i^B(\sigma) - C(x(i, \sigma)) &= v(\sigma, \tau_i) - v(\sigma_{i+1}, \tau_i) + \\ &+ \sum_{h=i+1}^T [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)] - C(x(i, \sigma)). \end{aligned} \quad (\text{A.22})$$

This expression, and therefore the net payoff $\pi_i^B(\sigma) - C(x(i, \sigma))$, is strictly concave in σ (by strict concavity of $v(\cdot, \tau_i)$, $\sigma(i, \cdot)$ and strict convexity of $C(\cdot)$) in the interval $(\sigma_{i+1}, \sigma_{i-1})$ and reaches a maximum at $\sigma_i = \sigma(i, x_i(i))$ as defined in (24) above.

Notice, further, that in the right adjoining interval $(\sigma_{i-1}, \sigma_{i-2})$, by Lemma 1 and Proposition

2, the net payoff $\pi_i^B(\sigma) - C(x(i, \sigma))$ takes the following expression — different from (A.22).

$$\begin{aligned} \pi_i^B(\sigma) - C(x(i, \sigma)) &= v(\sigma, \tau_{i-1}) - v(\sigma_{i-1}, \tau_{i-1}) + \\ &+ v(\sigma_{i-1}, \tau_i) - v(\sigma_{i+1}, \tau_i) + \\ &+ \sum_{h=i+1}^T [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)] - C(x(i, \sigma)). \end{aligned} \quad (\text{A.23})$$

This new expression of the net payoff $\pi_i^B(\sigma) - C(x(i, \sigma))$ is also strictly concave (by strict concavity of $v(\cdot, \tau_{i-1})$, $\sigma(i, \cdot)$ and strict convexity of $C(\cdot)$) and reaches a maximum at $\sigma(i, x_i(i-1))$. From Lemma 4 above we know that

$$\sigma(i, x_i(i-1)) < \sigma_{i-1} = \sigma(i-1, x_{i-1}(i-1)).$$

This implies that in the interval $(\sigma_{i-1}, \sigma_{i-2})$ the net payoff $\pi_i^B(\sigma) - C(x(i, \sigma))$ is strictly decreasing in σ .

A symmetric argument shows that the net payoff $\pi_i^B(\sigma) - C(x(i, \sigma))$ is strictly decreasing in σ in any interval (σ_h, σ_{h-1}) for every $h = 2, \dots, i-2$.

Notice, further, that in the left adjoining interval $(\sigma_{i+2}, \sigma_{i+1})$, by Lemma 1 and Proposition 2, the net payoff $\pi_i^B(\sigma) - C(x(i, \sigma))$ takes the following expression — different from (A.22) and (A.23).

$$\begin{aligned} \pi_i^B(\sigma) - C(x(i, \sigma)) &= v(\sigma, \tau_{i+1}) - v(\sigma_{i+2}, \tau_{i+1}) + \\ &+ \sum_{h=i+2}^T [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)] - C(x(i, \sigma)). \end{aligned} \quad (\text{A.24})$$

This new expression of the net payoff $\pi_i^B(\sigma) - C(x(i, \sigma))$ is also strictly concave in σ (by strict concavity of $v(\cdot, \tau_{i+1})$, $\sigma(i, \cdot)$ and strict convexity of $C(\cdot)$) and reaches a maximum at $\sigma(i, x_i(i+1))$ that from Lemma 4 is such that

$$\sigma_{i+1} = \sigma(i+1, x_{i+1}(i+1)) < \sigma(i, x_i(i+1)).$$

This implies that in the interval $(\sigma_{i+2}, \sigma_{i+1})$ the net payoff $\pi_i^B(\sigma) - C(x(i, \sigma))$ is strictly increasing in σ .

A symmetric argument shows that the net payoff $\pi_i^B(\sigma) - C(x(i, \sigma))$ is strictly increasing in σ in any interval (σ_{k+1}, σ_k) for every $k = i+2, \dots, T-1$. ■

Proof of Proposition 5: First, for a given ordered vector of sellers' qualities (τ_1, \dots, τ_T) we construct an inefficient equilibrium of the buyers' investment game such that there exist one buyer, labelled s_j , $j \in \{2, \dots, S\}$, such that $s_j < s_{j-1}$.

To show that a vector $(s_1, \dots, s_j, \dots, s_S)$ is an equilibrium of the buyers' investment game we need to verify that condition (27) holds for every $i = 2, \dots, S$ and no buyer s_i has an incentive to deviate and choose an investment x different from $x_{s_i}(i)$, as defined in (23).

Notice first that for every buyer, other than s_j and s_{j-1} , Proposition 4 above applies and hence it is an equilibrium for each buyer to choose investment level $x_{s_i}(i)$, as defined in (23), such that (27) is satisfied.

We can therefore restrict attention on buyer s_j and s_{j-1} . In particular we need to consider a buyer s_{j-1} of a quality arbitrarily close to the one of buyer s_j . This is achieved by considering a sequence of quality functions $\sigma^n(s_{j-1}, \cdot)$ that converges uniformly to $\sigma(s_j, \cdot)$.²¹ Then from definition (23), the continuity and strict concavity of $v(\cdot, \tau)$ and $\sigma(s, \cdot)$, the continuity and strict convexity of $C(\cdot)$ and the continuity of $v_1(\cdot, \tau)$, $\sigma_2(s, \cdot)$ and $C'(\cdot)$ for any given $\varepsilon > 0$ there exists an index n_ε such that from every $n > n_\varepsilon$:

$$|\sigma^n(s_{j-1}, x_{s_{j-1}}(j-1)) - \sigma(s_j, x_{s_j}(j-1))| < \varepsilon. \quad (\text{A.25})$$

From Lemma 4 and the assumptions $s_j > s_{j-1}$ we also know that for every $n > n_\varepsilon$:

$$\sigma^n(s_{j-1}, x_{s_{j-1}}(i-1)) < \sigma(s_j, x_{s_j}(j-1)). \quad (\text{A.26})$$

While from the assumption $\tau_j < \tau_{j-1}$ we have that:

$$\sigma(s_j, x_{s_j}(j)) < \sigma(s_j, x_{s_j}(j-1)). \quad (\text{A.27})$$

Inequalities (A.25), (A.26) and (A.27) imply that for any buyer s_{j-1} characterized by the quality function $\sigma^n(s_{j-1}, \cdot)$ where $n > n_\varepsilon$, the equilibrium condition (27) is satisfied:

$$\sigma(s_j, x_{s_j}(j)) < \sigma^n(s_{j-1}, x_{s_{j-1}}(j-1)). \quad (\text{A.28})$$

To conclude that $(s_1, \dots, s_j, \dots, s_S)$ is an equilibrium of the buyers' investment game we still need to show that neither buyer s_j nor buyer s_{j-1} want to deviate and choose an investment different from $x_{s_j}(j)$ and $x_{s_{j-1}}(j-1)$, where the quality function associated with buyer s_{j-1} is $\sigma^n(s_{j-1}, \cdot)$ for $n > n_\varepsilon$.

²¹The sequence $\sigma^n(s_{j-1}, \cdot)$ converges uniformly to $\sigma(s_j, \cdot)$ if and only if

$$\lim_{n \rightarrow \infty} \sup_x |\sigma^n(s_{j-1}, x) - \sigma(s_j, x)| = 0.$$

Consider the net payoff to buyer s_j : $\pi_{s_j}^B(\sigma) - C(x(s_j, \sigma))$. An argument symmetric to the one used in Step 2 of Proposition 4 shows that this payoff function is continuous in σ . Moreover, from the notation of σ_j in Section 4 above, Lemma 4, (A.26) and (A.28) we obtain that

$$\sigma_j < \sigma_{j-1}^n < \sigma(s_j, x_{s_j}(j-1)) < \sigma_{j-2}.$$

Then using an argument symmetric to the one used in Step 3 of the proof of Proposition 4 we conclude that this net payoff function has two local maxima at σ_j and $\sigma(s_j, x_{s_j}(j-1))$ and a kink at σ_{j-1}^n . We then need to show that there exist at least an element of the sequence σ_{j-1}^n such that the net payoff $\pi_{s_j}^B(\sigma) - C(x(s_j, \sigma))$ reaches a global maximum at σ_j . Therefore when the quality function of buyer s_{j-1} is $\sigma^n(s_{j-1}, \cdot)$ buyer s_j has no incentive to deviate and choose a different investment.

From (12) the net payoff $\pi_{s_j}^B(\sigma) - C(x(s_j, \sigma))$ computed at σ_j is greater than the same net payoff computed at $\sigma(s_j, x_{s_j}(j-1))$ if and only if

$$\begin{aligned} v(\sigma_j, \tau_j) - C(x(s_j, \sigma_j)) &\geq \\ &\geq v(\sigma(s_j, x_{s_j}(j-1)), \tau_{j-1}) - v(\sigma_{j-1}^n, \tau_{j-1}) + \\ &\quad + v(\sigma_{j-1}^n, \tau_j) - C(x(s_j, \sigma(s_j, x_{s_j}(j-1)))) \end{aligned} \quad (\text{A.29})$$

Inequality (A.25) above and the continuity of $v(\cdot, \tau_{j-1})$, $\sigma(s_j, \cdot)$ and $C(\cdot)$ imply that for any given $\varepsilon > 0$ there exist a ξ_ε and a n_{ξ_ε} such that for every $n > n_{\xi_\varepsilon}$

$$|v(\sigma(s_j, x_{s_j}(j-1)), \tau_{j-1}) - v(\sigma_{j-1}^n, \tau_{j-1})| < \xi_\varepsilon$$

and

$$|C(x(s_j, \sigma(s_j, x_{s_j}(j-1)))) - C(x(s_j, \sigma_{j-1}^n))| < \xi_\varepsilon$$

These two inequalities imply that a necessary condition for (A.29) to be satisfied is

$$v(\sigma_j, \tau_j) - C(x(s_j, \sigma_j)) \geq v(\sigma_{j-1}^n, \tau_j) - C(x(s_j, \sigma_{j-1}^n)) + 2\xi_\varepsilon. \quad (\text{A.30})$$

We can now conclude that there exist an $\varepsilon > 0$ such that for every $n > n_{\xi_\varepsilon}$ condition (A.30) is satisfied with strict inequality. This is because (by strict concavity of $v(\cdot, \tau_j)$, $\sigma(s_j, \cdot)$ and strict convexity of $C(\cdot)$) the function $v(\sigma, \tau_j) - C(x(s_j, \sigma))$ is strictly concave and has a unique interior maximum at σ_j .

Consider now the net payoff to buyer s_{j-1} : $\pi_{s_{j-1}}^B(\sigma) - C(x(s_{j-1}, \sigma))$. An argument symmetric to the one used above allow us to prove that this payoff function is continuous in σ . Further, from

the notation of σ_j in Section 4 above, Lemma 4, and (A.28) we have that

$$\sigma_{j+1} < \sigma^n(s_{j-1}, x_{s_{j-1}}(j)) < \sigma_j < \sigma_{j-1}^n.$$

Therefore we conclude that the net surplus function $\pi_{s_{j-1}}^B(\sigma) - C(x(s_{j-1}, \sigma))$ has two local maxima at σ_{j-1}^n and $\sigma^n(s_{j-1}, x_{s_{j-1}}(j))$ and a kink at σ_j . We still need to prove that there exist at least an element of the sequence σ_{j-1}^n such that the net payoff $\pi_{s_{j-1}}^B(\sigma) - C(x(s_{j-1}, \sigma))$ reaches a global maximum at σ_{j-1}^n which implies that when the quality function of buyer s_{j-1} is $\sigma^n(s_{j-1}, \cdot)$ this buyer has no incentive to deviate and choose a different investment.

From (12) the net payoff $\pi_{s_{j-1}}^B(\sigma) - C(x(s_{j-1}, \sigma))$ computed at σ_{j-1}^n is greater than the same net payoff computed at $\sigma^n(s_{j-1}, x_{s_{j-1}}(j))$ if and only if

$$\begin{aligned} v(\sigma_{j-1}^n, \tau_{j-1}) &- v(\sigma_j, \tau_{j-1}) + v(\sigma_j, \tau_j) - C(x(s_{j-1}, \sigma_{j-1}^n)) \geq \\ &\geq v(\sigma^n(s_{j-1}, x_{s_{j-1}}(j)), \tau_j) - C(x(s_{j-1}, \sigma^n(s_{j-1}, x_{s_{j-1}}(j)))) \end{aligned} \quad (\text{A.31})$$

Definition (23), the continuity and strict concavity of $v(\cdot, \tau_j)$ and $\sigma(s_{j-1}, \cdot)$, the continuity and strict convexity of $C(\cdot)$ and the continuity of $v_1(\cdot, \tau_j)$, $\sigma_2(s_j, \cdot)$ and $C'(\cdot)$ imply that for given $\varepsilon' > 0$ there exists a $n_{\varepsilon'}$, a $\xi_{\varepsilon'}$ and a $n_{\xi_{\varepsilon'}}$ such that from every $n > n_{\varepsilon'}$:

$$|\sigma^n(s_{j-1}, x_{s_{j-1}}(j)) - \sigma_j| < \varepsilon';$$

while for every $n > n_{\xi_{\varepsilon'}}$

$$|v(\sigma_j, \tau_j) - v(\sigma^n(s_{j-1}, x_{s_{j-1}}(j)), \tau_j)| < \xi_{\varepsilon'}$$

and

$$\left| C(x(s_{j-1}, \sigma_j)) - C(x(s_{j-1}, \sigma^n(s_{j-1}, x_{s_{j-1}}(j)))) \right| < \xi_{\varepsilon'}.$$

The last two inequalities imply that a necessary condition for (A.31) to be satisfied is

$$v(\sigma_{j-1}^n, \tau_{j-1}) - C(x(s_{j-1}, \sigma_{j-1}^n)) \geq v(\sigma_j, \tau_{j-1}) - C(x(s_{j-1}, \sigma_j)) + 2\xi_{\varepsilon'}. \quad (\text{A.32})$$

We can now conclude that there exists a $\varepsilon' > 0$ such that for every $n > n_{\xi_{\varepsilon'}}$ condition (A.32) is satisfied with strict inequality. This is because (by strict concavity of $v(\cdot, \tau_{j-1})$, $\sigma^n(s_{j-1}, \cdot)$ and strict convexity of $C(\cdot)$) the function $v(\sigma, \tau_{j-1}) - C(x(s_{j-1}, \sigma))$ is strictly concave and has a unique interior maximum at σ_{j-1}^n .

This concludes the construction of the inefficient equilibrium of the buyers' investment game.

We need now to show that for any given vector of buyers' quality functions $(\sigma(s_1, \cdot), \dots, \sigma(s_S, \cdot))$ it is possible to construct an ordered vector of sellers qualities (τ_1, \dots, τ_T) such that no inefficient

equilibrium exist.

Assume, by way of contradiction, that an inefficient equilibrium exists for any ordered vector of sellers' qualities (τ_1, \dots, τ_T) . Consider first the case in which this inefficient equilibrium is such that there exist only one buyer s_j such that $s_j < s_{j-1}$. Let τ_{j-1}^n be a sequence of quality levels of seller $(j-1)$ such that $\tau_{j-1}^n > \tau_j$ and τ_{j-1}^n converges to τ_j .

From Lemma 4 and the assumption $s_j > s_{j-1}$ we have that

$$\sigma(s_j, x_{s_j}(j)) > \sigma(s_{j-1}, x_{s_{j-1}}(j)) \quad (\text{A.33})$$

where $x_{s_j}(j)$ and $x_{s_{j-1}}(j)$ are defined in (23). Further, denote $x_{s_{j-1}}^n(j-1)$ the optimal investment defined, as in (24), by the following set of first order conditions:

$$v_1(\sigma(s_{j-1}, x_{s_{j-1}}^n(j-1)), \tau_{j-1}^n) \sigma_2(s_{j-1}, x_{s_{j-1}}^n(j-1), \tau_{j-1}^n) = C'(x_{s_{j-1}}^n(j-1)).$$

Then from Lemma 4 we have that

$$\sigma(s_{j-1}, x_{s_{j-1}}^n(j-1)) > \sigma(s_{j-1}, x_{s_{j-1}}(j)). \quad (\text{A.34})$$

Further, continuity of the functions $v(\sigma, \cdot)$, $v_1(\sigma, \cdot)$, $\sigma(s, \cdot)$, $\sigma_2(s, \cdot)$, $C(\cdot)$ and $C'(\cdot)$ imply that for given $\hat{\varepsilon} > 0$ there exist an $n_{\hat{\varepsilon}}$ such that for every $n > n_{\hat{\varepsilon}}$

$$\left| \sigma(s_{j-1}, x_{s_{j-1}}^n(j-1)) - \sigma(s_{j-1}, x_{s_{j-1}}(j)) \right| < \hat{\varepsilon}. \quad (\text{A.35})$$

Then from (A.33), (A.34) and (A.35) there exists an $\hat{\varepsilon} > 0$ and hence an $n_{\hat{\varepsilon}}$ such that for every $n > n_{\hat{\varepsilon}}$

$$\sigma(s_j, x_{s_j}(j)) > \sigma(s_{j-1}, x_{s_{j-1}}^n(j-1)). \quad (\text{A.36})$$

Inequality (A.36) clearly contradicts the necessary condition (27) for the existence of the inefficient equilibrium.

A similar construction leads to a contradiction in the case the inefficient equilibrium is characterized by more than one buyer s_j such that $s_j < s_{j-1}$. ■

Proof of Corollary 1: Notice first that the proofs of Lemma 3 and Lemma 4 hold unchanged in the case sellers choose their bids in the order of any vector of sellers' qualities (τ_1, \dots, τ_T) .

The proof of Proposition 4 also holds in this general case provided one substitutes the payoff in (12) with the payoffs in (10). Moreover we need to reinterpret the buyers' qualities σ_{i-1} , σ_i and σ_{i-1} to be the qualities of three subsequent buyers in the chain of runner-up buyers. In particular σ_i is the quality of the runner-up buyer to the seller that in equilibrium is matched with the buyer

of quality σ_{i-1} , while σ_{i+1} is the quality of the runner-up buyer to the seller that in equilibrium is matched with the buyer of quality σ_i . We do not repeat here the details of the proof.

Finally, the proof of Proposition 5 can also be modified to apply to the general case in which seller choose their bids in the order of the vector of sellers' qualities (τ_1, \dots, τ_T) . We need to substitute the payoff in (12) with the payoff in (10). Moreover, we need to reinterpret the buyer's identity s_j as the identity of the runner-up buyer to the seller that in equilibrium matches with the buyer s_{j-1} . Once again we do not repeat here the details of the proof. ■

Proof of Proposition 6: We prove this result in two steps. We first show that if sellers choose investments $y(t, t+1)$, for $t = 1, \dots, T$, (labelled *simple* investments, for convenience) then the order of sellers' identities coincides with the order of sellers' qualities. Hence, Proposition 2 applies and the shares of the surplus accruing to each buyer and each seller are the ones defined in (12) and (13) above. We then conclude the proof by showing that the unique equilibrium of the sellers' investment subgame is for seller t to choose the simple investment $y(t, t+1)$, $t = 1, \dots, T$.

Step 1. If each seller t chooses the simple investment $y(t, t+1)$, as defined in (28), then

$$\tau_1 = \tau(1, y(1, 2)) > \dots > \tau_T = \tau(T, y(T, T+1)).$$

The proof follows from the fact that from (32) we obtain:

$$\frac{\partial \tau(t, y(t, s))}{\partial t} = \frac{v_2 \tau_1 \tau_{22} - \tau_1 C'' - v_2 \tau_2 \tau_{12}}{v_{22}(\tau_2)^2 + v_2 \tau_{22} - C''} < 0 \quad (\text{A.37})$$

and

$$\frac{\partial \tau(t, y(t, s))}{\partial s} = \frac{v_{12}(\tau_2)^2}{v_{22}(\tau_2)^2 + v_2 \tau_{22} - C''} < 0 \quad (\text{A.38})$$

where (with an abuse of notation) we denote with τ_h and τ_{hk} , $h, k \in \{1, 2\}$ the first and second order derivatives of the quality functions $\tau(\cdot, \cdot)$ computed at $(t, y(t, s))$. Moreover the first and second order derivative (v_h and v_{hk} , $h, k \in \{1, 2\}$) of the functions $v(\cdot, \cdot)$ are computed at $(\sigma_s, \tau(t, y(t, s)))$ and C'' is evaluated at $y(t, s)$.

Step 2. The unique equilibrium of the sellers' investment subgame is such that seller t chooses the simple investment $y(t, t+1)$ for every $t = 1, \dots, T$.

We prove this result starting from seller T . In the T -th (the last) matching subgame of the Bertrand competition game all sellers, but seller T , have selected a buyer's bid. Denote τ_T the quality of this seller.

Assume for simplicity that $S = T + 1$. We use the same notation as in the proof of Proposition 2 above. In particular since we want to show that seller T chooses a simple investment independently from the investment choice of the other sellers we denote $\alpha_{(T)}$ and $\alpha_{(T+1)}$ the qualities of the two buyers that are still un-matched in the T -th subgame, such that $\alpha_{(T)} > \alpha_{(T+1)}$. Indeed, from Lemma 1 the identity of the two buyers left will depend on the order of sellers' qualities and therefore on the investment choices of the other $(T - 1)$ sellers.

From Lemma 1 above we have that the buyer of quality $\alpha_{(T)}$ matches with seller T . Seller T 's payoff is $v(\alpha_{(T+1)}, \tau_T)$ while the payoff of the buyer of quality $\alpha_{(T)}$ is $[v(\alpha_{(T)}, \tau_T) - v(\alpha_{(T+1)}, \tau_T)]$ and the payoff of the buyer of quality $\alpha_{(T+1)}$ is zero.

Denote now $a_{(T)}$, respectively $a_{(T+1)}$, the identity of the buyers of quality $\alpha_{(T)}$, respectively $\alpha_{(T+1)}$: $a_{(T)} < a_{(T+1)}$. Seller T 's optimal investment y_T is then defined as follows

$$y_T = \underset{y}{\operatorname{argmax}} v(\alpha(T + 1), \tau(T, y)) - C(y).$$

This implies that the optimal investment of seller T is the simple investment $y_T = y(T, a_{(T+1)})$, as defined in (32), whatever is the pair of buyers left in the T -th subgame. If all other sellers undertake a simple investment then from Step 1: $a_{(T)} = T$ and $a_{(T+1)} = T + 1$. Hence seller T 's optimal investment is $y(T, T + 1)$.

Denote now $t + 1$, ($t < T$), the last seller that undertakes a simple investment $y(t + 1, t + 2)$. We then show that also seller t will choose a simple investment $y(t, t + 1)$. Consider the t -th subgame in which seller t has to choose among the potential bids of the remaining $(T - t + 2)$ buyers labelled $a_{(t)} < \dots < a_{(T+1)}$, with associated qualities $\alpha_{(t)} > \dots > \alpha_{(T+1)}$, respectively.²² From the assumption that every seller $j = t + 1, \dots, T$ undertakes a simple investment $y(j, a_{(j+1)})$ and Step 1 we obtain that $\tau_{t+1} > \dots > \tau_T$. We first show that the quality associated with seller t is such that $\tau_t > \tau_{t+1}$.

Assume by way of contradiction that seller t chooses investment y^* that yields a quality τ^* such that $\tau_{j+1} \leq \tau^* \leq \tau_j$ for some $j \in \{t + 1, \dots, T - 1\}$. Then from Lemma 1 and (13) we have that seller t matches with buyer $a_{(j)}$ and seller t 's payoff is:

$$\Pi_{\tau^*}^S = v(\alpha_{(j+1)}, \tau(t, y^*)) - \sum_{h=j+1}^T [v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h)] \quad (\text{A.39})$$

²²Once again we want to show that seller t undertakes a simple investment independently of the investment choice of sellers $1, \dots, t - 1$ that, from Lemma 1, determines the exact identities of the un-matched buyers in the t -th subgame of the Bertrand competition game.

where $\tau(t, y^*) = \tau^*$. From (A.39) we obtain that y^* is then the solution to the following problem:

$$y^* = \operatorname{argmax}_y v(\alpha(j+1), \tau(t, y)) - C(y). \quad (\text{A.40})$$

From the assumption that each seller $j \in \{t+1, \dots, T\}$ undertakes a simple investment and definition (28) we also have that seller j 's investment choice $y(j, a_{(j+1)})$ is defined as follows:

$$y(j, a_{(j+1)}) = \operatorname{argmax}_y v(\alpha(j+1), \tau(j, y)) - C(y). \quad (\text{A.41})$$

Notice further that the payoff to seller t in (A.39) is continuous in τ^* . Indeed the limit for τ^* that converges from the right to τ_j is equal to

$$\Pi_{\tau_j}^S = v(\alpha_{(j+1)}, \tau_j) - \sum_{h=j+1}^T [v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h)]. \quad (\text{A.42})$$

If instead $\tau_j < \tau^* \leq \tau_{j-1}$ then from (13) the payoff to the seller with quality τ^* is

$$\begin{aligned} \Pi_{\tau^*}^S &= v(\alpha_{(j)}, \tau^*) - v(\alpha_{(j)}, \tau_j) + \\ &+ v(\alpha_{(j+1)}, \tau_j) - \sum_{h=j+1}^T [v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h)]. \end{aligned}$$

Therefore the limit for τ^* that converges to τ_j from the left is, from (8), equal to $\Pi_{\tau_j}^S$ in (A.42). This proves the continuity in τ^* of the payoff function in (A.39).

Continuity of the payoff function in (A.39) together with definitions (A.40), (A.41) and condition (A.37) imply that $y^* > y(j, a_{(j+1)})$ or $\tau^* > \tau_j$ a contradiction to the hypothesis $\tau^* \leq \tau_j$.

We now show that seller t will choose a simple investment $y(t, a_{(t+1)})$. From the result we just obtained $\tau_t > \tau_{t+1} > \dots > \tau_T$ and the assumption that $\alpha_{(t)} > \dots > \alpha_{(S)}$ are the qualities of the unmatched buyers in the t -th subgame of the Bertrand competition game we conclude, using (13) above, that the payoff to seller t is:

$$\Pi_{\tau_t}^S = v(\alpha_{(t+1)}, \tau_t) - \sum_{h=t+1}^T [v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h)] \quad (\text{A.43})$$

Seller t 's investment choice is then the simple investment $y(t, a_{(t+1)})$ defined as follows:

$$y(t, a_{(t+1)}) = \operatorname{argmax}_y v(\alpha(t+1), \tau(t, y)) - C(y). \quad (\text{A.44})$$

To conclude that a simple investment $y(t, a_{(t+1)})$ is the unique equilibrium choice for seller t in

the sellers' investment game we still need to show that seller t has no incentive to deviate and choose an investment y^* , and hence a quality τ^* , that exceeds the quality τ_k of one of the $(t - 1)$ sellers that are already matched at the t -th subgame of the Bertrand competition game: $k < t$. The reason why this choice of investment might be optimal for seller t is that it changes the pool of buyers $a_{(t)}, \dots, a_{(S)}$ unmatched in subgame t . Of course this choice will change the simple nature of seller t 's investment only if $\tau_k > \tau_{t+1}$. Indeed we already showed that if $\tau_k < \tau_{t+1}$ then $\tau_t > \tau_k$ and from (A.44) seller t 's investment choice is $y_t(a_{(t+1)})$ a simple investment for any given set of unmatched buyers.

Consider the following deviation by seller t : seller t chooses an investment $y^* > y(t, a_{(t+1)})$ that yields quality $\tau^* > \tau_k > \tau_{t+1}$. Recall that Lemma 1 implies that the ranking of each seller in the ordered vector of sellers' qualities determines the buyer each seller is matched with. Hence, seller t 's deviation changes the ranking and the matches of all sellers whose quality τ is smaller than τ^* and greater than τ_{t+1} . However, this deviation does not alter the ranking of the $T - t$ sellers with identities $(t + 1, \dots, T)$ and qualities $(\tau_{t+1}, \dots, \tau_T)$. Therefore, the only difference between the equilibrium set of un-matched buyers in the t -th subgame and the set of un-matched buyers in the same subgame following seller t 's deviation is the identity and quality of the buyer that matches with seller t .²³ The remaining set of buyers' identities and qualities $(\alpha_{(t+1)}, \dots, \alpha_{(S)})$ is unchanged.

Hence, following seller t 's deviation the un-matched buyers' qualities are $\alpha^* > \alpha_{(t+1)} > \dots > \alpha_{(T)}$, where α^* is the quality of the buyer that according to Lemma 1 is matched with seller t when the quality of this seller is τ^* . Equation (13) implies that seller t 's payoff following this deviation is then:

$$\Pi_{\tau^*}^S = v(\alpha_{(t+1)}, \tau^*) - \sum_{h=t+1}^T [v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h)] \quad (\text{A.45})$$

Continuity of the payoff function in (A.44) together with (A.45) imply that seller t 's net payoff is maximized at $y(t, a_{(t+1)})$. Hence, seller t cannot gain from choosing an investment $y^* > y(t, a_{(t+1)})$. This proves that seller t will choose a simple investment $y(t, a_{(t+1)})$. This argument holds for every $t < T$ implying that all seller choose a simple investment. Therefore $a_{(t)} = t$ and seller t 's equilibrium investment choice is $y_t = y(t, t + 1)$. ■

Proof of Proposition 8: Notice first that L and M can be written as

$$L = \sum_{t=1}^T \omega(t, t) - \sum_{t=1}^T \omega(t, t + 1) \quad (\text{A.46})$$

²³All other sellers with identities $(k, \dots, t - 1)$ whose match changed because of the deviation are already matched in the t -th subgame of the Bertrand competition game.

$$M = \sum_{t=1}^T \omega(1, t) - \sum_{t=1}^T \omega(1, t+1) \quad (\text{A.47})$$

so that

$$M - L = \sum_{t=1}^T \left\{ \left[\omega(1, t) - \omega(t, t) \right] - \left[\omega(1, t+1) - \omega(t, t+1) \right] \right\} \quad (\text{A.48})$$

From (A.48), it is clear that, as $T > 1$, each bracketed term in the summation will be positive with some strictly positive if

$$\frac{\partial^2 \omega(t, s)}{\partial s \partial t} > 0. \quad (\text{A.49})$$

From the definition (36) of $\omega(t, s)$ we have:

$$\frac{\partial^2 \omega(t, s)}{\partial s \partial t} = \frac{\partial}{\partial t} \left[(v_2 - \tilde{v}_2) \tau_2 \frac{\partial y(t, s)}{\partial s} \right]. \quad (\text{A.50})$$

Notice that from $v_{12}(\sigma, \tau) > 0$ we have $(v_2 - \tilde{v}_2) > 0$ if $s > t$; while from (32) we have:

$$\frac{\partial y(t, s)}{\partial s} = - \frac{\tilde{v}_{12} \sigma_1 \tau_2}{\tilde{v}_{22}(\tau_2)^2 + \tilde{v}_2 \tau_{22} - C''} < 0. \quad (\text{A.51})$$

From (A.50) the cross partial derivative of $\omega(t, s)$ then takes the following expression:

$$\begin{aligned} \frac{\partial^2 \omega(t, s)}{\partial s \partial t} &= \left[v_{12} + (v_{22} - \tilde{v}_{22}) \frac{\partial \tau(t, y(t, s))}{\partial t} \right] \tau_2 \frac{\partial y(t, s)}{\partial s} + \\ &+ (v_2 - \tilde{v}_2) \frac{\partial y(t, s)}{\partial s} \frac{\partial \tau_2(t, y(t, s))}{\partial t} + \\ &+ (v_2 - \tilde{v}_2) \tau_2 \frac{\partial^2 y(t, s)}{\partial s \partial t}. \end{aligned} \quad (\text{A.52})$$

To investigate the actual sign of $(\partial^2 \omega(t, s) / \partial s \partial t)$, we must identify the sign of $(v_{22} - \tilde{v}_{22})$, of the partial derivative $(\partial \tau_2(t, y(t, s)) / \partial t)$ and of the cross derivative $(\partial^2 y(t, s) / \partial s \partial t)$.

Notice first that the marginal complementarity assumption $v_{122} > 0$ implies that if $s > t$

$$(v_{22} - \tilde{v}_{22}) > 0. \quad (\text{A.53})$$

Second, from the definition (32) of $y(t, s)$ we have that:

$$\frac{\partial \tau_2(t, y(t, s))}{\partial t} = \frac{\tau_{12} \tilde{v}_{22} (\tau_2)^2 - C'' \tau_{12} - \tau_{22} \tilde{v}_{22} \tau_1 \tau_2}{\tilde{v}_{22} (\tau_2)^2 + \tilde{v}_2 \tau_{22} - C''} < 0. \quad (\text{A.54})$$

In expressions (A.50), (A.51), (A.52), (A.53) and (A.54) the derivatives v_h and v_{hk} , $h, k \in \{1, 2\}$, are evaluated at $(\sigma(t), \tau(t, y(t, s)))$, while \tilde{v}_h and \tilde{v}_{hk} , $h, k \in \{1, 2\}$, are evaluated at $(\sigma(s), \tau(t, y(t, s)))$, and τ_h and τ_{hk} , $h, k \in \{1, 2\}$, are evaluated at $(t, y(t, s))$. Finally, C'' is evaluated at $y(t, s)$.

The responsive complementarity assumption (33) implies that:

$$\frac{\partial^2 y(t, s)}{\partial s \partial t} > 0. \quad (\text{A.55})$$

Conditions (A.53), (A.54) and (A.55) imply — together with (A.51) and $(v_2 - \tilde{v}_2) > 0$ if $s > t$ — that all three terms in (A.52) are strictly positive. Thus $(\partial^2 \omega(t, s) / \partial s \partial t)$ is positive: every term in the summation of (A.48) is positive and $M > L$. ■

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