EC487 Advanced Microeconomics, Part I: Lecture 2

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LSE economics

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Recall the following property of the profit function

\[ \pi(p, w) = \max_x p f(x) - w x = p f(x(p, w)) - w x(p, w) \]

- **Hotelling’s Lemma:**

\[ \frac{\partial \pi}{\partial p} = y(p, w) \geq 0 \quad \text{and} \quad \frac{\partial \pi}{\partial w_i} = -x_i(p, w) \leq 0 \]
Consider now the Hessian matrix of the profit function $\pi(p, w)$:

$$
H = \begin{pmatrix}
\frac{\partial^2 \pi}{\partial p^2} & \frac{\partial^2 \pi}{\partial p \partial w_1} & \ldots & \frac{\partial^2 \pi}{\partial p \partial w_h} \\
\frac{\partial^2 \pi}{\partial w_1 \partial p} & \frac{\partial^2 \pi}{\partial w_1^2} & \ldots & \frac{\partial^2 \pi}{\partial w_1 \partial w_h} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 \pi}{\partial w_h \partial p} & \frac{\partial^2 \pi}{\partial w_h \partial w_1} & \ldots & \frac{\partial^2 \pi}{\partial w_h^2}
\end{pmatrix}
$$

By Hotelling’s Lemma the matrix $H$ is:

$$
H = \begin{pmatrix}
\frac{\partial y}{\partial p} & \frac{\partial y}{\partial w_1} & \ldots & \frac{\partial y}{\partial w_h} \\
-\frac{\partial x_1}{\partial p} & -\frac{\partial x_1}{\partial w_1} & \ldots & -\frac{\partial x_1}{\partial w_h} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{\partial x_h}{\partial p} & -\frac{\partial x_h}{\partial w_1} & \ldots & -\frac{\partial x_h}{\partial w_h}
\end{pmatrix}
$$
Result

Let $F : A \rightarrow \mathbb{R}$ where $A \subset \mathbb{R}^n$ is a convex open set. Let $F(\cdot)$ be twice differentiable. Then the function $F(\cdot)$ is convex if and only if the Hessian matrix of $F(\cdot)$ is positive semi-definite on $A$.

- Recall that the profit function $\pi(p, w)$ is convex in $(p, w)$.

- This together with Young theorem implies that $H$ is symmetric and has a non negative main diagonal.

- Assume now that $\pi(p, w)$ is defined on an open and convex set $S$ of prices $(p, w)$ then convexity of $\pi(p, w)$ is equivalent to the Hessian Matrix $H$ being positive semi-definite.
Profit maximization can be achieved in two sequential steps:

1. Given $y$, find the choice of inputs that allows the producer to obtain $y$ at the minimum cost;

   this generates *conditional factor demands* and the *cost function*;

2. Given the cost function, find *the profit maximizing output level*. 
Notice: step 1 is common to firms that behave *competitively* in the input market but *not necessarily* in the output market.

Only in step 2 we impose the *competitive assumption on the output market.*
We shall start from step 1:

\[
\begin{align*}
\min_{x} & \quad w \times \nabla f(x) \\
\text{s.t.} & \quad f(x) \geq y
\end{align*}
\]

The necessary *first order* conditions are:

\[
y = f(x^*),
\]

\[
w \geq \lambda \nabla f(x^*)
\]

and

\[
[w - \lambda \nabla f(x^*)] \times x^* = 0
\]
or for every input $\ell = 1, \ldots, h$:

$$w_\ell \geq \lambda \frac{\partial f(x^*)}{\partial x_\ell}$$

with equality if $x^*_\ell > 0$.

The first order conditions are also sufficient if $f(x)$ is quasi-concave (the input requirement set is convex).

Alternatively, a set of sufficient conditions for a local minimum are that $f(x)$ is quasi-concave in a neighborhood of $x^*$. 
This can be checked (sufficient condition) by means of the bordered hessian matrix and its minors.

In the case of only two inputs $f(x_1, x_2)$ we have:

$$w_\ell \geq \lambda \frac{\partial f(x^*)}{\partial x_\ell}, \quad \forall \ell = 1, 2$$

with equality if $x_\ell^* > 0$

and SOC:

$$\begin{vmatrix}
0 & f_1(x^*) & f_2(x^*) \\
 f_1(x^*) & f_{11}(x^*) & f_{12}(x^*) \\
 f_2(x^*) & f_{21}(x^*) & f_{22}(x^*)
\end{vmatrix} > 0$$
In the case the two first order conditions are satisfied with equality (no corner solutions) we can rewrite the necessary conditions as:

\[ \text{MRTS} = \left| \frac{dx_2}{dx_1} \right| = \frac{\partial f(x^*)/\partial x_1}{\partial f(x^*)/\partial x_2} = \frac{w_1}{w_2} \]

and

\[ y = f(x^*) \]

Notice a close formal analogy with consumption theory (expenditure minimization).
This leads to define:

- the solution to the cost minimization problem:

\[ x^* = z(w, y) = \begin{pmatrix} z_1(w, y) \\ \vdots \\ z_h(w, y) \end{pmatrix} \]

- the \textit{conditional (to } y \textit{) factor demands} (correspondences).

- the minimand function of the cost minimization problem:

\[ c(w, y) = w \cdot z(w, y) \]

- the \textit{cost function}.
Properties of the Cost Function

1. $c(w, y)$ is non-decreasing in $y$.

**Proof:** Suppose not. Then there exist $y' < y''$ such that (denote $x'$ and $x''$ the corresponding solution to the cost minimization problem)

$$w \times' \geq w \times'' > 0$$

If the latter inequality is strict we have an immediate contradiction of $x'$ solving the cost minimization problem.
If on the other hand
\[ w \ x' = w \ x'' > 0 \]

then by continuity and monotonicity of \( f(\cdot) \) there exists \( \alpha \in (0, 1) \) close enough to 1 such that
\[ f(\alpha \ x'') > y' \]

and
\[ w \ x' > w \ \alpha x'' \]

which contradicts \( x' \) solving the cost minimization problem.
2. \( c(w, y) \) is non-decreasing with respect to \( w_\ell \) for every \( \ell = 1, \ldots, h \).

**Proof:** Consider \( w' \) and \( w'' \) such that \( w''_\ell \geq w'_\ell \) but \( w''_k = w'_k \) for every \( k \neq \ell \).

Let \( x'' \) and \( x' \) be the solutions to the cost minimization problem with \( w'' \) and \( w' \) respectively.

Then by definition of \( c(w, y) \)

\[
c(w'', y) = w'' x'' \geq w' x'' \geq w' x' = c(w', y). \qed
\]
Properties of the Cost Function (cont’d)

3. \( c(w, y) \) is homogeneous of degree 1 in \( w \).

**Proof:** The feasible set of the cost minimization problem

\[
f(x) \geq y
\]

does not change when \( w \) is multiplied by the factor \( t > 0 \).

Hence \( \forall t > 0 \), minimizing \( (t \, w) \, x \) on this set leads to the same answer as minimizing \( w \, x \).

Let \( x^* \) be the solution, then:

\[
c(t \, w, y) = (t \, w) \, x^* = t \, c(w, y).
\]
4. \( c(w, y) \) is a concave function in \( w \).

**Proof:** Let \( \hat{w} = t \, w + (1 - t) \, w' \) for \( t \in [0, 1] \).

Let \( \hat{x} \) be the solution to the cost minimization problem for \( \hat{w} \). Then

\[
c(\hat{w}, y) = \hat{w} \, \hat{x} = t \, w \, \hat{x} + (1 - t) \, w' \, \hat{x}
\]

\[
\geq t \, c(w, y) + (1 - t) \, c(w', y)
\]

by definition of \( c(w, y) \) and \( c(w', y) \) and \( f(\hat{x}) \geq y \).
5. **Shephard’s Lemma:**

If \( z(w, y) \) is single valued with respect to \( w \) then \( c(w, y) \) is differentiable with respect to \( w \) and

\[
\frac{\partial c(w, y)}{\partial w} = z_\ell(w, y)
\]

**Proof:** By constrained Envelope Theorem and the definition of cost function

\[
c(w, y) = w \cdot z(w, y) = w \cdot z(w, y) - \lambda(w, y) [f(z(w, y)) - y]
\]
6. \( z(w, y) \) is homogeneous of degree 0 in \( w \).

**Proof:** By Shepard’s Lemma and the following result.

**Result**

\[ \text{If a function } G(x) \text{ is homogeneous of degree } r \text{ in } x \text{ then } \left( \frac{\partial G}{\partial x_\ell} \right) \text{ is homogeneous of degree } (r - 1) \text{ in } x \text{ for every } \ell = 1, \ldots, L. \]

**Proof:** Differentiate with respect to \( x_\ell \) the identity that defines homogeneity of degree \( r \):

\[ G(k \cdot x) \equiv k^r G(x) \quad \forall k > 0 \]
We obtain:

\[ k \frac{\partial G(k \, x)}{\partial x_\ell} = k^r \frac{\partial G(x)}{\partial x_\ell} \quad \forall k > 0 \]

or

\[ \frac{\partial G(k \, x)}{\partial x_\ell} = k^{r-1} \frac{\partial G(x)}{\partial x_\ell} \quad \forall k > 0 \]

This is the definition of homogeneity of degree \((r - 1)\) of the function \(\frac{\partial G(k \, x)}{\partial x_\ell}\).
Notice that

6. The Lagrange multiplier of the cost minimization problem is the marginal cost of output:

$$\frac{\partial c(w, y)}{\partial y} = \lambda(w, y)$$

**Proof:** By constrained Envelope Theorem applied to the cost function:

$$c(w, y) = w z(w, y) - \lambda(w, y) [f(z(w, y)) - y]$$
7. Let the set $W$ of input prices $w$ be open and convex, if $z(w, y)$ is differentiable in $w$ then:

$$H = \begin{pmatrix}
\frac{\partial^2 c}{\partial w_1^2} & \cdots & \frac{\partial^2 c}{\partial w_1 \partial w_h} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 c}{\partial w_h \partial w_1} & \cdots & \frac{\partial^2 c}{\partial w_h^2}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial z_1}{\partial w_1} & \cdots & \frac{\partial z_1}{\partial w_h} \\
\vdots & \ddots & \vdots \\
\frac{\partial z_h}{\partial w_1} & \cdots & \frac{\partial z_h}{\partial w_h}
\end{pmatrix}$$

is a *symmetric* and *negative semi-definite* matrix.
Law of Supply

**Proof:** Symmetry follows from Shephard’s lemma and Young Theorem:

\[
\frac{\partial z_\ell}{\partial w_i} = \frac{\partial}{\partial w_i} \left( \frac{\partial c(w, y)}{\partial w_\ell} \right) = \frac{\partial}{\partial w_\ell} \left( \frac{\partial c(w, y)}{\partial w_i} \right) = \frac{\partial z_i}{\partial w_\ell}
\]

While negative semi-definiteness follows from the concavity of \(c(w, y)\), the fact that \(W\) is open and convex and the following result.

**Result**

Let \(F : A \rightarrow \mathbb{R}\) where \(A \subset \mathbb{R}^n\) is a convex open set. Let \(F(\cdot)\) be twice differentiable. Then the function \(F(\cdot)\) is concave if and only if the Hessian matrix of \(F(\cdot)\) is negative semi-definite on \(A\).
Euler Theorem

Result (Euler Theorem)

*If a function $G(x)$ is homogeneous of degree $r$ in $x$ then:*

$$r \ G(x) = \nabla G(x) \ x$$

**Proof:** Differentiating with respect to $k$ the identity:

$$G(k \ x) \equiv k^r \ G(x) \quad \forall k > 0$$

we obtain:

$$\nabla G(kx) \ x = rk^{(r-1)} \ G(x) \quad \forall k > 0$$

for $k = 1$ we obtain:

$$\nabla G(x) \ x = r \ G(x).$$
We now introduce a set of new properties closely related to the ones of the expenditure function in consumer theory.

8. If $f(x)$ is homogeneous of degree one (i.e. exhibits constant returns to scale), then $c(w, y)$ and $z(w, y)$ are homogeneous of degree one in $y$.

**Proof:** Let $k > 0$ and consider:

$$c(w, k y) = \min_x w x$$
$$\text{s.t. } f(x) \geq k y$$

By definition of $c(w, y)$ if $x^*$ is the solution to the cost minimization problem we have $y = f(x^*)$. 

Hence by homogeneity of degree 1 of $f(x)$ we obtain:

$$k \ y = k \ f(x^*) = f(k \ x^*)$$

which implies that $k \ x^*$ is \textit{feasible} in Problem (1).

Therefore:

$$k \ c(w, y) = k \ [w \ x^*] = w \ (k \ x^*) \geq c(w, k \ y).$$
Let now $\hat{x}$ be the solution to Problem (1). Necessarily:

$$f(\hat{x}) = ky$$

or, by homogeneity of degree 1:

$$f[(1/k) \hat{x}] = (1/k) f(\hat{x}) = y$$

which implies that $[(1/k) \hat{x}]$ is feasible in the problem that defines $c(w, y)$.

Therefore we get:

$$c(w, ky) = w \hat{x} = kw [(1/k) \hat{x}] \geq k c(w, y)$$

which concludes the proof.
8'. In other words, a technology that exhibits CRS has a cost function that is linear in $y$: $c(w, y) = c(w)y$.

9. A technology that exhibits CRS has equal and constant marginal $(\partial c(w, y)/\partial y)$ and average cost functions

$$(\partial c(w, y)/\partial y) = (c(w, y)/y).$$
Proof: Homogeneity of degree 1 and Euler theorem imply
\[ c_y(w)y = c(w, y) \] or \[ c_y(w) = \frac{c(w, y)}{y}. \]
10. If $f(x)$ is convex (IRS technology), then the cost function $c(w, y)$ is concave in $y$.

11. A technology that exhibits IRS has a decreasing marginal cost function $\left( \frac{\partial c(w, y)}{\partial y} \right)$ and average cost function $\left( \frac{\partial c(w, y)}{\partial y} \right) \leq \left( \frac{c(w, y)}{y} \right)$
Returns to Scale (cont’d)

\[ \frac{c(w, y)}{y} \]

\[ c_y(w, y) \]

\[ c(w, y) \]
12. If $f(x)$ is concave (DRS technology), then the cost function $c(w, y)$ is convex in $y$.

13. A technology that exhibits DRS has an increasing marginal cost function \( \frac{\partial c(w, y)}{\partial y} \) and average cost function \( \frac{\partial c(w, y)}{\partial y} \geq \frac{c(w, y)}{y} \)
Returns to Scale (cont’d)

\[ c_y(w, y) \]

\[
\frac{c(w, y)}{y}
\]

\[ c(w, y) \]
Assume that the output market is competitive.

The profit maximization problem is then:

\[
\max_y \quad p y - c(w, y)
\]

The necessary **FOC** are:

\[
p - \frac{\partial c(w, y^*)}{\partial y} \leq 0
\]

with equality if \( y^* > 0 \).

The sufficient **SOC** for a local maximum:

\[
\frac{\partial^2 c(w, y^*)}{\partial y^2} > 0
\]
Clearly the SOC imply at least local DRS in a neighborhood of $y^*$.  

Notice that if $y^* > 0$ the optimal choice of the firm is:

$$ p = \frac{\partial c(w, y^*)}{\partial y} = MC(y^*) $$

In words, price equal to marginal cost.

This condition defines the solution to the profit maximization problem: the supply function: $y^*(w, p)$.
The two profit maximization problems we have considered so far produce the same outcome for equal \((w, p)\).

In fact:

\[
\max_y \quad p y - c(w, y)
\]

where

\[
c(w, y) = \min_x \quad w x \quad \text{s.t.} \quad f(x) \geq y
\]

yields

\[
\max_x \quad p y - w x \quad \text{s.t.} \quad f(x) = y
\]

which is the very first problem we considered.
Short run cost minimization arises when one or more inputs may be fixed, $x_h = \bar{x}_h$, while the remaining inputs may be varied at will.

The short run variable cost function:

$$c^S(w, y, \bar{x}_h) = w_h \bar{x}_h + \min_{x_1, \ldots, x_{h-1}} \sum_{\ell=1}^{h-1} w_\ell x_\ell$$

s.t. $f(x_1, \ldots, x_{h-1}, \bar{x}_h) \geq y$

Alternatively:

$$c^S(w, y, \bar{x}_h) = \min_x w x$$

s.t. $f(x) \geq y$

$x_h = \bar{x}_h$
If \( z(w, y) \) denotes the long run conditional factor demands, that solve:

\[
c(w, y) = \min_x w \cdot x \\
\text{s.t. } f(x) \geq y
\]

Let \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_h) \) be the input vector that achieves the minimum long run cost of producing \( \bar{y} \) with prices \( \bar{w} \):

\[
\bar{x} = (\bar{x}_1, \ldots, \bar{x}_h) = z(\bar{w}, \bar{y})
\]
Short Run vs. Long Run

We can then characterize the *relationship between short and long run total costs*, or alternatively, short run and long run variable costs (more familiar).

Notice that

\[ c(w, y) \equiv c^S(w, y, z_h(w, y)) \]  \hspace{1cm} (2)

or

\[ \frac{c(w, y)}{y} \equiv \frac{c^S(w, y, z_h(w, y))}{y} \]
moreover by Envelope Theorem

\[ \frac{\partial c(w, y)}{\partial y} \equiv \frac{\partial c^S(w, y, z_h(w, y))}{\partial y} \]  

(3)

We shall now focus on a neighborhood of \((\bar{w}, \bar{y})\) and set \(\bar{x}_h = z_h(\bar{w}, \bar{y})\). From (2) above by Envelope Theorem we get:

\[ \frac{\partial c(w, y)}{\partial y} = \frac{\partial c^S(w, y, \bar{x}_h)}{\partial y} \]

Recall that Envelope Theorem implies that only the \textit{first order effect} is zero.
Since (3) is an identity in \((w, y)\) we can differentiate both sides with respect to \(y\):

\[
\frac{\partial^2 c^S(w, y, \bar{x}_h)}{\partial y^2} + \frac{\partial^2 c^S(w, y, \bar{x}_h)}{\partial y \partial \bar{x}_h} \frac{\partial z_h(w, y)}{\partial y} = \frac{\partial^2 c(w, y)}{\partial y^2}
\]

and with respect to \(w_h\):

\[
\frac{\partial^2 c^S(w, y, \bar{x}_h)}{\partial y \partial w_h} + \frac{\partial^2 c^S(w, y, \bar{x}_h)}{\partial y \partial \bar{x}_h} \frac{\partial z_h(w, y)}{\partial w_h} = \frac{\partial^2 c(w, y)}{\partial y \partial w_h}
\]
Now
\[
\frac{\partial^2 c^S(w, y, \bar{x}_h)}{\partial y \partial w_h} = 0
\]

since
\[
\frac{\partial c^S(w, y, \bar{x}_h)}{\partial w_h} = \bar{x}_h
\]
is independent of $y$.

Hence by Shephard’s Lemma:
\[
\frac{\partial^2 c^S(w, y, \bar{x}_h)}{\partial y \partial \bar{x}_h} = \frac{\partial z_h(w, y)}{\partial y} / \frac{\partial z_h(w, y)}{\partial w_h}
\]
which implies by substitution:

\[
\frac{\partial^2 c^S(w, y, \bar{x}_h)}{\partial y^2} + \left( \frac{\partial z_h(w, y)/\partial y}{\partial z_h(w, y)/\partial w_h} \right)^2 = \frac{\partial^2 c(w, y)}{\partial y^2}
\]

which delivers:

\[
\frac{\partial^2 c^S(w, y, \bar{x}_h)}{\partial y^2} \geq \frac{\partial^2 c(w, y)}{\partial y^2}
\]

since

\[
\frac{\left( \frac{\partial z_h(w, y)/\partial y}{\partial z_h(w, y)/\partial w_h} \right)^2}{\partial z_h(w, y)/\partial w_h} \leq 0
\]
This allows us to conclude that the loss function:

\[ l(w, y) = c(w, y) - c^S(w, y, \bar{x}_h) \leq 0 \]

reaches a local maximum at \( \bar{x} \).

**Proof:** We have shown above that the FOC are satisfied:

\[
\frac{\partial c^S(w, y, \bar{x}_h)}{\partial y} = \frac{\partial c(w, y)}{\partial y}
\]

and we just proved that the SOC hold:

\[
\frac{\partial^2 c(w, y)}{\partial y^2} \leq \frac{\partial^2 c^S(w, y, \bar{x}_h)}{\partial y^2}
\]
Le Chatelier Principle

In other words:

\[ c(w, y) \leq c^S(w, y, \bar{x}_h) \]

A similar approach proves that for every \( \ell \in \{1, \ldots, h\} \):

\[ 0 \geq \frac{\partial z^S_{\ell}}{\partial w_{\ell}} \geq \frac{\partial z_{\ell}}{\partial w_{\ell}} \]

Moving to profit maximization:

\[ 0 \geq \frac{\partial x^S_{\ell}}{\partial w_{\ell}} \geq \frac{\partial x_{\ell}}{\partial w_{\ell}} \]

and

\[ 0 \leq \frac{\partial y^S}{\partial p} \leq \frac{\partial y}{\partial p} \]

All these results are known as *Le Chatelier Principle*. 
The question we address is when can we speak of *an aggregate supply function* in similar terms as the aggregate demand you have seen before?

The question above is closely related to the question: under which conditions can we speak of a *representative producer*?

Recall that the key problem when constructing an aggregate demand was the presence of income effects in consumer theory.
The absence of a budget constraint implies that individual firms’ supply are not subject to income effects.

Hence aggregation of production theory is simpler and requires less restrictive conditions.

Consider $J$ production technologies: $(Z^1, \ldots, Z^J)$

Let $z^j(p, w) = \left( \begin{array}{c} -x^j(p, w) \\ y^j(p, w) \end{array} \right)$ be firm $j$’s production plan.
We have seen that the matrix of cross and own price effects of production plan $z^j(p,w)$:

$$Dz^j(p,w)$$

is symmetric and positive semi-definite.

This is also known as law of supply.
Aggregate Law of Supply

Define now the following aggregate optimal production plan:

\[ z(p, w) = \sum_{j=1}^{J} z^j(p, w) = \left( -\frac{\sum_j x^j(p, w)}{\sum_j y^j(p, w)} \right) \]

Does an aggregate law of supply hold?

Since both symmetry and positive semi-definiteness are preserved under sum then

\[ Dz(p, w) \]

is also symmetric and positive semi-definite.
Representative Producer

Result

In a purely competitive environment the maximum profit obtained by every firm maximizing profits separately is the same as the profit obtained if all J firms where they coordinate their choices in a joint profit maximization:

\[ \pi(p, w) = \sum_{j=1}^{J} \pi^j(p, w) \]

In other words, there exists a representative producer.