Competing Premarital Investments

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This paper studies premarital parental investments in children’s wealth, where spousal wealth is a public good in marriage. By investing in their children’s wealth, parents increase the wealth of their children and the quality of the spouses that their children can marry. In large marriage markets, the hedonic return to investment internalizes all the external benefits of premarital investment in wealth so that the competitive equilibrium is efficient. Marriage market competition also increases investments in small marriage markets relative to no competition, but equilibrium investments are not efficient.

I. Introduction

In marriage, an individual derives utility from own premarital investment and the premarital investment of his or her spouse. Much of these investments are human capital investments made by altruistic parents. Since premarital investment is a public good in marriage, parents may underinvest in their children.1

This literature typically assumes that the child’s marriage partner is fixed and independent of any investment that the family makes in the

1 For a discussion of the incentive for private provision of public goods, see Bergstrom, Blume, and Varian (1986) and references cited therein. More specifically, MacLeod and Malcomson (1993) explore investment incentives in bilateral matching problems and, in particular, discuss the impact of outside options on these incentives. For a discussion of investment incentives in marriage with outside options but without marriage market competition, see Konrad and Lommerud (2000).

We thank the referees, seminar participants at the University of Toronto, State University of New York at Buffalo, and participants in the Canadian Economic Theory Conference for their useful comments. We also gratefully acknowledge financial support from the Social Sciences and Humanities Research Council. Siow also acknowledges an intellectual debt to Sherwin Rosen, whose lectures 20 years ago resonate here.
child. In this sense the arguments leave out the important effects that competition for spouses will have on ex ante investment incentives. In this paper, we study premarital investments when children use these investments to compete for spouses. We are primarily interested in the implications of assortative matching equilibria, which occur when wealthy individuals are matched with wealthy partners. Then altruistic parents take into account the additional utility their children will enjoy from wealthier partners, and this will increase their incentive to invest in their children on the margin.

Our first model considers the case in which the number of families is very large. We study a competitive equilibrium in which all families on the same side of the market believe that they face the same nonstochastic return to their investment in their children. This return function adjusts until families’ beliefs are fulfilled in equilibrium.

Perhaps the most remarkable property of investment in the competitive equilibrium is the fact that the externalities associated with families’ investments in their children are completely internalized by this return function. Any pair of families whose children match on the competitive equilibrium path will make investments that are bilaterally Pareto optimal. Despite the fact that neither family can directly compensate the other family for the investment that it makes in its child, the marriage market and the assortative matching that occurs there force each family to compensate the other indirectly through the investment that it makes in its own child.

This efficiency result is an application of Rosen’s (1974) hedonic pricing approach to large matching problems. The market return function provides what is essentially a hedonic value for every investment level that a family might consider making. Families on the other side of the market need to provide these hedonic values in order to attract partners with specific investment levels. In equilibrium, each family’s indifference curve (in the space of investments) will be tangent to this hedonic return function, and consequently, families whose children match will have indifference curves that are tangent to each other. The investments that families undertake will then be bilaterally efficient in the sense that there will not be another pair of investments that will make both of the matched families better off at the same time. Since the joint payoffs that we employ are supermodular (Becker 1973; Smith 1996), assortative matching and bilateral efficiency are sufficient to guarantee that the distribution of investments for the economy is efficient.

Matters are more complicated in small marriage markets. When the number of families and children is small, assortative matching among children will raise families’ incentive to invest and at the same time make families’ investments less predictable. If there are significant wealth disparities on the other side of the market, parents may find that
they can increase the wealth of their child’s partner significantly by raising their investment only slightly. This makes parental payoff functions discontinuous, which rules out pure strategy equilibria in some situations. In the mixed-strategy equilibria that do prevail, parental investment is stochastic. Though rich families will invest more in their children, on average, than poor families do, there will be a positive probability that the poor families will invest more than rich families so that their children move up the wealth distribution. This creates endogenous intergenerational mobility.

The small-numbers case is perhaps not so interesting in the context of the family matching problem in which large market arguments seem quite natural. However, our methods apply to a variety of other bilateral matching markets in which small numbers are more important. For example, workers and firms face a similar difficulty with ex ante investments in physical and human capital, even though a wider variety of contractual remedies for holdup problems are available in that context. Some problems in which small numbers are likely to be more important are coauthorship in academics, sports teams, business partnerships, or other forms of collaborative activity. We have not been able to provide a complete link between the small- and large-numbers cases, but we can give complete characterization of equilibrium for a special case that illustrates most of the issues involved.

Ex ante investments in labor markets have been analyzed in a couple of papers. In an interesting paper, Shi (1997) allows firms to choose which of several market segments to join and then to pick a capital investment optimal for the workers who are available in that segment. Since the workers’ human capital endowments are exogenous and matching is random within a segment, his results are not easy to compare with ours. The primary difference is that workers and firms cannot move across market segments once their investments have been made, so workers do not compete against one another for partners and the matching return to investment that we describe does not arise.

Closer to our treatment is a paper by Felli and Roberts (2000) in which workers offer to work for firms at wages that depend on their human capital investments. Each worker offers a wage that makes its target firm just indifferent between accepting the offer and accepting the offer of the next most skilled worker. The logic of their equilibrium is reminiscent of the logic used to show that bidders bid their true valuations in second-price auctions. The profit a worker needs to offer a firm is determined by the human capital investment of a worker that the firm does not hire on the equilibrium path. The worker that the firm does hire gets all the residual and so has the correct ex ante investment incentives. Their procedure differs from ours in two ways. First, utility is transferable and the return to investment is a monetary
payment that the worker extracts from the firm instead of a matching return. Though their procedure resolves the holdup problem for workers, it does not achieve full efficiency since firms’ investment incentives are not inefficient.

The closest work to ours is that by Cole, Mailath, and Postlewaite (2001a, 2001b). They consider a family matching problem with transfers in which investments are followed by a cooperative matching process that selects partners and redistributes income across children. They show that efficient investments can be supported as an equilibrium outcome, though other inefficient equilibria are also possible. Efficient investments occur because the cooperative matching process generates an explicit monetary return to investment. In our model there is no monetary return to investment, and marginal incentives are created solely by the expectation that higher investment will improve match quality.

Han (2001) extends the worker-firm investment matching model to environments in which traders disagree about who the most desirable partners are. In addition, he provides a survey and comparison of the equilibrium outcomes of the papers listed above.

Siow and Zhu (in press) also study a large marriage market and premarital investment problem with transferable utility and two wealth classes on each side of the market. They also study multigenerational equilibria. Acemoglu (1997) studies a two-side matching investment model with workers and firms. He obtains underinvestment because, owing to potential random matching, workers and firms are unable to fully capture the returns to their pre-employment investments.

While this paper focuses on the marriage market, our analysis applies to other partnerships in which the share of surplus in the partnership is not conditioned on the level of prepartnership investment. Members of amateur sports teams and coauthors in economics usually do not divide surplus according to their levels of prepartnership investments. In most of these markets, agents invest in prepartnership human capital and then compete for partners. The results in this paper should be useful for thinking about those markets as well.

II. Preliminaries

Families begin with an endowment of wealth $y$ that can be used partly as current consumption and partly as an investment in children. Let $w$ be the amount invested in the child. If the child subsequently matches with a partner whose wealth level is $\tilde{w}$, then utility for the parents is given by

$$V(y - w) + z \cdot (w + \tilde{w}), \quad z > 0,$$
and utility of the child is \( z \cdot (w + \bar{w}) \). If the family invests \( w \) and the child is not expected to match, then we assume that the child has utility \( zw \). The actual value of the children’s utility when no match occurs is unimportant as long as both the child and family are at least weakly better off when a match occurs than if it does not.²

Assumption. The function \( V(\cdot) \) is monotonically increasing, strictly concave, and differentiable, has bounded marginal utilities, and satisfies

\[
\lim_{\epsilon \to 0} V(\epsilon) = -\infty.
\]

The bilateral Nash or noncompetitive investment levels for each family \( i \) are given by the solutions to

\[
V'(y, -w^*) = z.
\]

These are the investment levels that the families would make if they believed (for whatever reason) that their children’s match partner is independent of parental investment. In the case in which there are only two families, one on each side of the market, the investment of the family on the other side of the market would be fixed and equilibrium investment would satisfy equation (1). Investment would be inefficient in this case because (1) does not take account of the positive effect that the family’s investment has on the family on the other side of the market.

After the families have made their investments, the children compete for partners in the marriage market. We shall first study the investment and matching problem in a large marriage market. Then we shall investigate properties of small marriage markets.

III. Large Marriage Markets

This section considers a large marriage market with a continuum of families on each side of the market. We refer to families with female children as families “in \( F \)” and, similarly, families with male children as families “in \( M \).” Let the total measure of female families be \( F \) and the total measure of male families be \( M \), where \( F \geq M \). Let \( G \) and \( H \) be measures of sets of families in \( M \) and \( F \), respectively. Interpret \( G(B) \) to be the measure of the set of families whose endowments lie in the set \( B \), and similarly for \( H \). Suppose that the wealth levels of families in \( M \) are distributed on a closed interval \( Y \), whereas families in \( F \) have their wealth distributed on a closed interval \( \bar{Y} \).

Let \( g(w) \) represent the wealth of the wife that each family in \( M \) expects to match with from an investment of \( w \) in its son. If families’ expectations

² It is possible that children might strictly prefer not to match if the best available partner is too poor. This creates problems for our methodology, but they are not particularly relevant for the issues we wish to discuss.
are realized, then $g^{-1}(\tilde{w})$ will represent the wealth of the groom that each family in $F$ expects to match with from an investment of $\tilde{w}$ in its daughter. Since higher wealth attracts better partners with assortative matching, $g(w)$ will be nondecreasing in $w$.

**Definition 1.** The return function $g(w)$ is a rational expectations equilibrium if there exist investment strategies $\sigma(y)$ and $\tilde{\sigma}(\tilde{y})$ for families in $M$ and $F$, respectively, such that (1) $\sigma(y) \in \arg \max_y \{V(y-x) + z[x + g(x)]\}$ for each $y \in Y$ and $\tilde{\sigma}(\tilde{y}) \in \arg \max_{\tilde{y}} \{V(\tilde{y}-x) + z[x + g^{-1}(x)]\}$ for each $\tilde{y} \in \tilde{Y}$; (2) for each $w$, $G(y : \sigma(y) \geq w) = H(\tilde{y} : \tilde{\sigma}(\tilde{y}) \geq g(w))$.

Part 1 in the definition says that each family optimally chooses a wealth level for its child given the return function $g(w)$. Part 2 is the marriage market–clearing condition. It requires that the measure of the set of families in $M$ that (optimally) invest $w$ or more is exactly equal to the measure of the set of families in $F$ that invest $g(w)$ or more. By assortative matching, this means that every family in $M$ that invests $w$ or more will be able to find a partner for its child whose investment is $g(w)$ or more.

Figure 1 illustrates a rational expectations equilibrium. The investment levels for families in $M$ are given along the horizontal axis, and investments for families in $F$ are along the vertical axis. The dark curve illustrates the equilibrium matching function. The lighter curves that are convex upward are indifference curves for families in $M$, and those that are convex downward are indifference curves for families in $F$.

A family in $M$ that invests $w^{**}$ should expect its child to match with someone whose wealth is $g(w^{**})$. In equilibrium, if a family chooses investment $w^{**}$, then its indifference curve should be tangent to the curve $g(w)$ at the point $(w^{**}, g(w^{**}))$. The reason is that the family thinks that $g(w)$ represents the market return function that it faces. Similarly, any family in $F$ that chooses to invest $g(w^{**})$ should expect return $w^{**}$. In equilibrium, this family must have an indifference curve tangent to the market trade-off function $g^{-1}(\cdot)$ at the point $(w^{**}, g(w^{**}))$. This implies that the indifference curves of the families of every pair of children who match in equilibrium will be tangent to each other. So every pair of families that match will choose investments that are bilaterally Pareto optimal: the underinvestment problem disappears.

As the picture is drawn, the family from $M$ that attains the indifference curve $II$ is the one with the lowest endowment, whereas the family from $F$ that attains the indifference curve $II'$ touching the market return line at point $A$ is the family with the lowest endowment that actually succeeds in matching. In equilibrium, this family will have to be just indifferent between making a positive investment and matching with the poorest family in $M$, and investing Nash and having no partner at all. That is why the indifference curve $II'$ just touches the vertical axis at $w$.

To see this more formally, focus on the case in which both $G$ and $H$ are monotonic, and let $\alpha(y)$ satisfy $M - G(y) = F - H(\alpha(y))$. By assor-
tative matching, a family in $M$ with income $y$ should end up matching with a family from $F$ whose income is $\alpha(y)$ provided that each family's investment is an increasing function of its endowment. A necessary condition for optimality is that a family from $M$ whose income is $y$ should prefer to invest $\sigma(y)$ rather than to invest $\sigma(y')$, as they would if their income were $y'$ instead of $y$. In equilibrium with assortative matching, each family's payoff can be written as

$$V(y - \sigma(y)) + z[\sigma(y) + \tilde{\alpha}(\alpha(y))].$$

This can be no smaller than $V(y - \sigma(y')) + z[\sigma(y') + \tilde{\alpha}(\alpha(y'))].$ So differ-
entiating this latter expression with respect to $y'$ and setting the resulting derivative evaluated at $y' = y$ equal to zero gives the necessary condition

$$\sigma'(y)[-V'(y - \sigma(y)) + z] = -\tilde{\sigma}'(\alpha(y))\alpha'(y).$$  \hspace{1cm} (2)

The corresponding condition for the family $\alpha(y)$ from $F$ is that

$$-z\sigma'(y)\frac{1}{\alpha'(y)} = \tilde{\sigma}'(\alpha(y))[-V'(\alpha(y) - \sigma(\alpha(y))) + z].$$  \hspace{1cm} (3)

Dividing (2) by (3) gives equality of the marginal rates of substitution between own investment and partner wealth (i.e., tangency of the families’ indifference curves in fig. 1 above).

This tangency condition means that there is no way that a pair of families whose children match can jointly change their investments to make both families better off. There remains only the possibility that families could be rematched and then adjust their investments in a way that will make everyone better off. The payoff function we employ has a simple single crossing property that rules this out. So an immediate consequence of bilateral efficiency is the following proposition.

**Proposition 1.** A rational expectations equilibrium is Pareto optimal.

*Proof.* We show that if families are rematched in a nonassortative fashion, then at least one family must be made worse off than it is in the rational expectations equilibrium. Figure 1 can be used to illustrate this. Suppose that a Pareto improvement can be obtained by rematching in such a way that family $A$ in $M$ (i.e., the family that ends up at allocation $A$ in fig. 1) is rematched with family $B$ from $F$. Family $A$ is the family in figure 1 that chooses point $A$, whereas family $B$ is the family from $F$ that chooses the outcome $B$. If matching family $B$ with family $A$ results in a Pareto improvement, then investment levels need to be adjusted so that both families end up on higher indifference curves than they attain in the initial assortative matching equilibrium. Since family $B$ chooses a higher investment level than $A$’s original partner, family $B$ must have higher wealth than $A$’s initial partner by assortative matching, so $B$’s indifference curve will be too steep to cross family $A$’s indifference curve and a Pareto-improving change in investment levels is impossible. Q.E.D.

IV. Existence of a Rational Expectations Equilibrium

Our notion of rational expectations equilibrium can be applied no matter what the properties of the underlying distributions $G$ and $H$ happen to be. For example, $G$ and $H$ could consist of atoms or agglomerations
of families at specific wealth levels. When a rational expectations equilibrium exists, it will be Pareto optimal as argued above. In some simple problems, existence of a rational expectations equilibrium is immediate. For example, suppose that \( G = H \). Then set \( g(w) = w \). Then each family chooses the (efficient) level of investment \( w^* \) that equates the marginal utility of consumption and \( 2z \). For a variety of reasons, this is not a good example of the rational expectations solution concept. We shall return to it soon.

A more illuminating example occurs when \( G \) and \( H \) differ. Suppose as before that \( F > M \) but that all the families in \( F \) have the same endowment. In figure 1, suppose that all families in \( F \) have indifference curves like \( II' \). Define \( g(x) \) so that it coincides with \( II' \). Let the families in \( M \) choose the points on this indifference curve that they most desire. Families in \( F \) can then choose investment levels to match with families in \( M \) along this indifference curve. The market return function will have a closed-form solution provided that the indifference curve can be represented in closed form.

To avoid making the mechanics too complicated, we sketch the argument for existence of equilibrium in the case in which the distribution functions \( G \) and \( H \) are both monotonic with differentiable inverse functions whose derivatives are bounded away from zero and infinity. Define for each \( y \) in the support of \( G 
\end{equation}

Since \( G \) and \( H \) are both monotonic and differentiable, so is \( \gamma \). Furthermore, the derivative of \( \gamma \) is bounded away from zero and infinity. The first-order condition for the optimal investment for a family of income \( y \) is given by

\begin{equation}
-V'(y - w) + z[1 + g'(w)] = 0.
\end{equation}

Since \( V' \) is monotonic, it has an inverse. This implies that the income of the family in \( M \) that invests \( w \) must be equal to

\begin{equation}
V^{-1}(z[1 + g'(w)]) + w.
\end{equation}

The family in \( F \) that invests \( g(w) \) has an income level such that \( g(w) \) satisfies

\begin{equation}
-V'(y' - g(w)) + z\left[1 + \frac{1}{g'(w)}\right] = 0.
\end{equation}

\(^3\) One particularly simple case to imagine is the one in which all families on the same side of the market are identical. For example, in fig. 1, all families in \( M \) could have indifference curves given by \( II \), whereas all families in \( F \) have indifference curves given by \( II' \). The rational expectations return function drawn in that diagram would still work and provide an efficient rational expectations equilibrium.
In equilibrium, this family matches with the family in $M$ that invests $w$. If $g()$ is a rational expectations solution, the measure of families that are wealthier than this family from $M$ must be equal to the measure of families that are wealthier than $y'$. This requires that

$$y' = \gamma(V^{-1}(z[1 + g'(w)]) + w).$$

This yields the ordinary differential equation

$$-V'(\gamma(V'^{-1}(z[1 + g'(w)]) + w) - g(w)) + \left[1 + \frac{1}{g'(w)}\right] = 0.$$}

Rearranging gives

$$g'(w) = \gamma(V'^{-1}(z[1 + g'(w)]) + w)) - V'^{-1}\left(1 + \frac{1}{g'(w)}\right).$$

Let $\phi(g', w)$ denote the expression on the right-hand side of this equation. Since $\phi$ is monotonically decreasing in $g'$, the inverse function $\phi^{-1}(\cdot, w)$ exists. So the market return function must satisfy

$$g'(w) = \phi^{-1}(g(w), w).\quad (4)$$

To see the initial condition, let $y^*$ be the poorest family from $F$ that successfully matches with some family from $M$ in equilibrium ($y^*$ satisfies $F - H(y^*) = M$). Let $I^*$ be the highest indifference curve that the family with endowment $y^*$ can attain when it does not match. Let $(w, g(w))$ be the point at which this indifference curve is tangent to the indifference curve for a family in $M$ whose endowment is $y$. The equilibrium market return function is then any solution to (4) with initial condition $(w, g(w))$. To ensure existence and uniqueness of the solution, we need to impose additional restrictions on the problem to ensure that $\phi^{-1}$ satisfies the usual Lipshitz condition in $g$. We do not pursue these issues here.

V. A Hedonic Pricing Interpretation

The model above can be interpreted as a special case of Rosen’s hedonic market model. To see this most easily, consider families in $M$ as suppliers. Let $y$ be the characteristic of a supplier. Let $w$ be the level of output (premarital investment) that a supplier produces. Note that $w$ also provides consumption value for the supplier and thus is not purely costly. The function $g(w)$ is the return that a son gets for supplying $w$. We may consider families in $F$ as demanders. If a demander pays $\tilde{w}$, the daughter will match with a supplier whose output is $g^{-1}(\tilde{w})$. In contrast to Rosen’s model, demanders value paying $\tilde{w}$. However, this does not cause any analytic difficulty because a demander that is matched will pay a higher
than she is willing to pay if she is not matched. So as in the case of
Rosen’s firms, the demander will prefer to pay less \( \bar{w} \) for her matched
supplier if she could.

The existence of a market-clearing wealth-matching function \( g(w) \) is
useful for researchers who want to study parental investment decisions
in the face of marriage market considerations. For an individual family, \( g(w) \) summarizes the marriage market opportunities that family will face.
For example, Botticini and Siow (1999) assume the existence of such
a function to study the demand for dowries by individual families.

VI. Small Marriage Markets

We continue to assume without further discussion that children match
assortatively in investment during the matching process. It is not hard
to construct noncooperative matching games in which this occurs. One
approach is to follow the approach associated with wage-posting games
(e.g., Shi 1997) in which one side publicly announces the wealth that
it brings to a match, the other side proposes to the partner it most
prefers, and then the respondents pick their favorite partner among
those who have proposed to them. We do not pursue these details here.

It is also possible to show that the investment game among families
that occurs prior to this assortative matching process always has at least
one equilibrium that typically involves mixed strategies. The details of
this argument can be found in our working paper (Peters and Siow
2001).

We focus instead on the richest example in which the equilibrium
can be fully characterized. We do this for two reasons. First, this example
is of independent interest in matching problems such as coauthorship
or business partnership in which the number of investors on each side
of the market really is small. Second, this example illustrates the diffi-
culties associated with connecting the rational expectations solution with
exact equilibria in the family matching problem when the number of
families is large but finite.

We consider the case in which there are four families. One of the
families from \( F \) and one from \( M \) have an initial endowment \( y \), and the
other pair of families each have endowment \( y' \). Consider investment
levels \( w_f, w'_f, w_m, \text{ and } w'_m \), where in an obvious notation the superscript
\( f \) refers to children of families from \( F \) and the superscript \( m \) refers to
children of families in \( M \). In the case in which the investment levels
differ in the sense that, for example, \( w'_f > w_f \), the continuation equilib-
rium is straightforward. The wealthy child from \( F \) will propose to the
wealthy child from \( M \) with probability one, and this proposal will be
accepted (similarly for the less wealthy children). The poor \( F \) has no
incentive to propose to the wealthy \( M \) because she expects the wealthy
To propose there with probability one and she knows that the wealthy male will always prefer her proposal.

If the wealth disparity of the families is large enough, it will never pay families to try to improve their children’s match quality. In that case there will be an obvious equilibrium in which families make noncooperative investments and never mix. To avoid this, we make the following assumption.

Assumption.

\[ V(y_i - w_i^*) + z \cdot 2w_i^* > V(y_i - w_i^*) + z \cdot 2w_i^*. \]

The assumption says that a poor family would be willing to raise investment to \( w_i^* \) if it believed that this would generate a match with a child from a wealthy family making the noncooperative investment.

Then we have the following theorem.

Theorem 1. There exists a symmetric mixed-strategy equilibrium in which the wealthy families both use mixture \( H_h \) and the poor families both use mixture \( H_l \). These mixtures have the following properties: (1) \( H_l \) has an atom at \( w_i^* \) and \( H_h \) has an atom at \( w_i^* \), with \( H_l(w_i^*) < H_h(w_i^*) \); and (2) \( H_l \) and \( H_h \) both have smooth density functions on some interval \([w_i^*, w^*] \), with \( H_l(s) < H_h(s) \) for all \( s \in [w_i^*, w^*] \).

The first part of the proof of this theorem (in the Appendix) is constructive. The argument resembles the argument for a Bertrand competition with capacity constraints. If one family invests more than the other, that family’s child will match with the wealthiest child on the other side of the market. In the mixed-strategy equilibrium, investments generate random returns since the investment level of the competing family is random. We choose the supports and distributions for the mixed strategies so that each family is indifferent to every investment level in the support of its equilibrium strategy. The major complication involved in this argument arises from the fact that the family that invests most gets a random return equal to the first order statistic for investment levels on the other side of the market. The distribution of this order statistic is endogenous. So this apparently straightforward problem requires a fixed-point argument to give a complete result. The details of this part of the proof can be found in our working paper (Peters and Siow 2001).

The theorem illustrates nicely the inherent unpredictability of investment. Both kinds of families choose their bilateral Nash investment level with positive probability. However, they also use a strategy that involves investment at a level strictly above \( w_i^* \) with positive probability. It follows immediately that both families will invest, on average, strictly more than their bilateral Nash investment levels. From the position of the atoms and the restriction on densities, it is immediate that the family with the highest endowment will invest more, on average, than the family
with the low endowment. With randomization, however, there is a strictly positive probability that the poor family will end up investing more than the wealthy family. Its child will then be at the top of the wealth distribution. We interpret this as endogenous intergenerational mobility in wealth.

We have explored exact equilibria with more families, but apart from the general existence of mixed-strategy equilibria, it is difficult to provide a good characterization of Nash equilibrium in investment. In contrast to similar problems with discontinuous payoffs (e.g., Allen and Hellwig 1986), the payoff associated with being first is itself endogenous. This makes it difficult to see what restrictions on payoffs are required to keep the equilibrium mixed strategies well behaved.

VII. Discussion

There is also a close connection between the models in this paper and the directed search models of the labor market (e.g., Moen 1997; Shi 1999). To see this, suppose that there is a measurable set of firms that invest in physical capital and workers who invest in human capital. Firms have different technologies parameterized by some variable \( y \in \mathbb{R} \) with the marginal product of capital increasing in \( y \). Workers differ according to a parameter that determines the cost at which the worker can acquire human capital. This cost is assumed to be decreasing as \( y' \) increases. Each firm has a single job to be filled, and each worker wishes to fill one job. The total output produced by the firm is some increasing function of the physical capital \( w \) invested by the firm and the human capital \( w' \) invested by the worker who fills the job. Physical capital is purchased by the firm at a fixed price \( r \), and human capital is acquired by the worker according to a convex and increasing cost \( c(w') \). For the moment, assume that when a firm and worker match, each receives a fixed share of the profit that is created. So if a firm that invests physical capital \( w \) is matched with a worker with human capital \( w' \), the profit of the firm is

\[
\alpha f(w, w'; y) - rw,
\]

and the profit of the worker is

\[
(1 - \alpha)f(w, w') - c(w'; y').
\]

The functional form used in this example differs slightly from that used in the marriage market above, but otherwise the problems are identical. If we allow the firms to advertise their capital stocks after they make their investments so that workers can apply to the firm that they like, all the equilibria will involve assortative matching exactly as in the marriage problem.
The hedonic value of the firm’s investment $w$ is given by some function $g(w)$ that gives the human capital that will be embodied in the worker whom the firm expects to be able to attract. Conversely, any worker who wants a job at a firm with physical capital $w$ will have to provide the level $g(w)$ of human capital to get the job. In equilibrium, this hedonic value will ensure that firms and workers will invest efficiently. This is similar to the result in Moen (1997), though it generalizes that result by allowing firms and workers to differ and by endogenizing the investments on both sides of the market. There are also some important differences. In the existing literature on directed search in labor markets, frictions generated by workers’ inability to coordinate their search decisions play an important role. The hedonic value of any given wage that a firm offers to pay workers is then measured by the size of the queue of applicants that the firm attracts. The model here shows that when families or workers differ in equilibrium, the mixed-strategy equilibria that support these frictions disappear. The matching equilibria that occur after wages are posted or capital stocks are chosen involve pure assortative matching. Families use their own characteristics to coordinate their search decisions. Despite this, the hedonic interpretation in which the market responds to specific investments with a predictable return is supported.

The other major difference is that there are no side payments in the model studied here (in the labor market interpretation, firms do not offer wages but instead simply give workers an exogenously determined split of the profit). The case in which workers and firms have multidimensional characteristics is certainly likely to support a hedonic interpretation, but so far models of this form have not been studied.

One of the predictions of the model studied here is certainly too strong: pure assortative matching. Clearly the model needs to be extended to allow for unobservable or match-specific characteristics. The payoff to focusing on the case with perfect information is the simplicity of the model that it delivers. A synthesis of the directed search models of the kind discussed here and the random matching models that characterize the older literature is clearly an important topic for future research.

Appendix

Proof of Theorem 1

Restrict attention for the moment to families in $M$. The proof is constructive. Let $\bar{w}$ and $\bar{w}$ be the ex post expected levels of investment of the poorest and

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4 Frictions are generated by the fact that workers use mixed strategies when they choose which firms to apply to (Shi 1999; Peters 2000).
wealthiest child in $F$. Let $H_l$ and $H_h$ denote the probability mixtures used by the wealthy and poor families in $M$, respectively, and suppose that they satisfy the hypothesis of the theorem. In other words, $H_l$ and $H_h$ have atoms at $w^*$ and $w^*_h$, respectively, and are otherwise smooth on some interval $[w^*, w^{**}]$.

Consider the wealthy family first. If it turns out that its child has the highest investment level ex post, then he will match with the wealthiest child in $F$, who will have expected investment $\bar{w}$. If he is poorest ex post, he will match with the poorest child in $F$, gaining a partner whose expected wealth is $\underline{w}$. The only tie we need to worry about occurs when the poor family invests exactly $\bar{w}$ and this is matched by the rich family. In this case we assume that the wealthy child from $F$ proposes to the male from the poor family for sure. Under these conditions, the expected payoff when the family invests $w \in [w^*, w^{**}]$ is

$$H_l(w)[V(y_l - w) + z \cdot (\bar{w} + w)] + [1 - H_l(w)][V(y_l - w) + z \cdot (\underline{w} + w)].$$  

(A1)

To support the equilibrium, this must be constant along $[w^*, w^{**}]$ and equal to

$$V(y_l - w^*) + z \cdot (w^* + [1 - H_l(w^*)]V(y_l - w^*) + z \cdot \bar{w})$$

to induce the wealthy family in $M$ to make the investment. If the function is constant, its derivative should be almost everywhere zero, or

$$-V'(y_l - w) + z + H_l(w)z \cdot (\bar{w} - w) = 0,$$

which gives

$$H_l'(w) = \frac{V(y_l - w) - z}{z \cdot (\bar{w} - w)}.$$  

The function $H_l$ is then determined (up to a constant) by integrating

$$H_l(w) - H_l(w^*) = \int_{w^*}^w \frac{V(y_l - s)ds - z}{z \cdot (\bar{w} - w)}ds$$

$$= \frac{V(y_l - w^*) - V(y_l - w) - z \cdot (w - w^*)}{z \cdot (\bar{w} - w)},$$  

(A2)

where $H_l(w^*)$ is the probability that the low-wealth family invests $w^*_l$. This atom and the value of $w^{**}$ are determined below.

The poor family faces a similar problem. Let $H_h$ be the distribution of investments by the wealthy family. The poor family’s payoff is

$$H_h(w)[V(y_h - w) + z \cdot (w + \bar{w})] + [1 - H_h(w)][V(y_h - w) + z \cdot (w + \underline{w})],$$

and this should be constant on the interval $[w^*_h, w^{**}]$ and equal to

$$H_h(w^*_h)[V(y_h - w^*_h) + z \cdot (w^*_h + \bar{w})]$$

$$+ [1 - H_h(w^*_h)][V(y_h - w^*_h) + z \cdot (w^*_h + \underline{w})],$$  

(A3)

where $H_h(w^*_h)$ is the probability that the wealthy family invests $w^*_h$.

The poor family will choose the investment level $w^*_h$ on the equilibrium path.

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5 This is for notational convenience only; our results are unaffected by the tie-breaking rule. If the rich child from $F$ randomizes in some fashion, the payoff to playing $w^*_h$ exactly will still be smaller than the limit of the payoffs associated with playing slightly more than $w^*_h$, and this is all that is required to support our equilibrium.
so the atom in \( H_l(w^*) \) should be chosen to make the poor family indifferent between the investment levels \( w^* \) and \( w^+ \) conditional on the assumption that if the poor family invests \( w^* \) it will match with the rich family on the other side of the market in the event of ties. To accomplish this, assign the atom \( H_l(w^*) \) so that

\[
H_l(w^*)[V(y_i - w^*) + z \cdot (w^* + \bar{w})] + [1 - H_l(w^*)][V(y_i - w^*) + z \cdot (w^* + \bar{w})] = V(y_i - w^*) + z \cdot (w^* + \bar{w})
\]

or

\[
H_l(w^*) = \frac{V(y_i - w^*) + zw^* - V(y_i - w^*) - zw^*}{z \cdot (\bar{w} - w)}. \quad (A4)
\]

Then reasoning as above, we have that

\[
H_h(w^*) = \frac{V(y_i - w^*) + zw^* - V(y_i - w^*) - zw^*}{z \cdot (\bar{w} - w)}.
\]

Finally, there can be no atoms at the top of the distribution of investments because of the discontinuous increase in expected wealth that this creates. So the atom at \( H_l(w^*) \) (i.e., the probability that the poor family chooses investment \( w^* \)) should be chosen so that the tops of the supports of \( H_l \) and \( H_h \) coincide.

The top of the support of \( H_h \) is given by the solution to

\[
V(y_i - w^*) - V(y_i - w^{**}) - z \cdot (w^{**} - w^*)
\]

\[
= z \cdot (\bar{w} - w).
\]

This determines

\[
H_h(w^*) = 1 - \frac{V(y_i - w^*) - V(y_i - w^{**}) - z \cdot (w^{**} - w^*)}{z \cdot (\bar{w} - w)}. \quad (A7)
\]

Note that since the density of the distribution \( H_h \) is uniformly higher on the interval \([w^*, w^{**}]\) than the density of \( H_h \) it follows that the atom \( H_h(w^*) \) is strictly larger than the atom \( H_h(w^*) \), which verifies the two properties of the distributions mentioned in the theorem. One implication of this is that the mean investment of the wealthy family exceeds the mean investment of the poor family.

Conditional on the mean payoff levels \( \bar{w} \) and \( \bar{w} \), it is straightforward to show that neither family can profitably deviate from this strategy. The poor family is indifferent between investing \( w^* \) and any investment level in the support \([w^*, w^{**}]\) by construction. Investment levels between \( w^* \) and \( w^{**} \) guarantee a match with a partner whose expected wealth is \( \bar{w} \). Since this outcome is the same for every investment level on the interval \([w^*, w^{**}]\), the poor family’s expected utility is strictly higher when it invests \( w^* \) than when it invests any amount in \((w^*, w^{**})\) by the strict concavity of \( V \). Similarly, the quality of the poor family’s match is independent of its investment level if it tries to invest more.
References