Microeconomics II Lecture 3

Constrained Envelope Theorem

Consider the problem:

$$\max_x f(x)$$

s.t. $g(x, a) = 0$

The Lagrangian is:

$$L(x, \lambda, a) = f(x) - \lambda g(x, a)$$

Necessary FOC are:

$$f'(x^*) - \lambda^* \frac{\partial g(x^*, a)}{\partial x} = 0$$

$$g(x^*(a), a) = 0$$
Substituting $x^*(a)$ and $\lambda^*(a)$ in the Lagrangian we get:

$$L(a) = f(x^*(a)) - \lambda^*(a) \ g(x^*(a), a)$$

Differentiating we get:

$$\frac{dL(a)}{da} = \left[ f'(x^*) - \lambda^* \frac{\partial g(x^*, a)}{\partial x} \right] \frac{dx^*(a)}{da} - g(x^*(a), a) \frac{d\lambda^*(a)}{da} - \lambda^*(a) \frac{\partial g(x^*, a)}{\partial a}$$

$$= -\lambda^*(a) \frac{\partial g(x^*, a)}{\partial a}$$

by the necessary FOC.

In other words — to the first order — only the direct effect of $a$ on the Lagrangian function matters.
3. Roy’s identity:

\[ x_i(p, m) = -\frac{\partial V/\partial p_i}{\partial V/\partial m} \]

By the constrained envelope theorem and the observation that:

\[ V(p, m) = u(x(p, m)) - \lambda(p, m) \left[ p \ x(p, m) - m \right] \]

we shall obtain:

\[ \frac{\partial V}{\partial p_i} = -\lambda(p, m) x_i(p, m) \leq 0 \]

and

\[ \frac{\partial V}{\partial m} = \lambda(p, m) \geq 0 \]

which is the marginal utility of income.
(Notice that the sign of the two inequalities above prove property 1 of the indirect utility function $V(p, m)$.)

We conclude the proof substituting

$$\frac{\partial V}{\partial m} = \lambda(p, m)$$

into

$$\frac{\partial V}{\partial p_i} = -\lambda(p, m) \ x_i(p, m)$$

and solving for $x_i(p, m)$. ■

4. _Adding up_ results. From the identity:

$$p \ x(p, m) = m \quad \forall p, \quad \forall m$$
Differentiation with respect to $p_j$ gives:

$$x_j(p, m) + \sum_{i=1}^{L} p_i \frac{\partial x_i}{\partial p_j} = 0$$

or, more interestingly, with respect to $m$ gives:

$$\sum_{i=1}^{L} p_i \frac{\partial x_i}{\partial m} = 1$$

There does not exist a clear cut comparative-static property with the exception of:

$$0 \geq \sum_{i=1}^{L} p_i \frac{\partial x_i}{\partial p_h} = -x_h(p, m)$$

which means that at least one of the Marshallian demand function has to be downward sloping in $p_h$. 
Effect of a change in income on the level of the Marshallian demand:

$$\frac{\partial x_l}{\partial m}$$

In the two commodities graph the set of tangency points for different values of $m$ is known as the *income expansion path*.

In the commodity income graph the set of optimal choices of the quantity of the commodity is known as *Engel curve*.
We shall classify commodities with respect to the effect of changes in income in:

- **normal goods:**
  \[
  \frac{\partial x_l}{\partial m} > 0
  \]

- **neutral goods:**
  \[
  \frac{\partial x_l}{\partial m} = 0
  \]

- **inferior goods:**
  \[
  \frac{\partial x_l}{\partial m} < 0
  \]

Notice that for every level of income \( m \) at least one of the \( L \) commodities is normal:

\[
\sum_{l=1}^{L} p_l \frac{\partial x_l}{\partial m} = 1
\]
If the Engel curve is \textit{convex} we are facing a \textit{luxury good} in other case a \textit{necessity}.
Expenditure Minimization Problem

The dual problem of the consumer’s utility maximization problem is the expenditure minimization problem:

\[
\begin{align*}
\min_{\{x\}} \quad & p x \\
\text{s.t.} \quad & u(x) \geq U
\end{align*}
\]

Define the solution as:

\[
x = h(p, U) = \left( \begin{array}{c}
h_1(p_1, \ldots, p_L, U) \\
\vdots \\
h_L(p_1, \ldots, p_L, U)
\end{array} \right)
\]

the Hicksian (compensated) demand functions.

We shall also define:

\[
e(p, U) = p \ h(p, U)
\]

as the expenditure function.
Properties of the expenditure function:

1. Continuous in \( p \) and \( U \).

2. \( \frac{\partial e}{\partial U} > 0 \) (2.1) and \( \frac{\partial e}{\partial p_l} \geq 0 \) (2.2) for every \( l = 1, \ldots, L \).

Proof: (2.1): Suppose not: there exist \( U' < U'' \) (denote \( x' \) and \( x'' \) the corresponding solution to the e.m.p.) such that \( p \ x' \geq p \ x'' > 0 \).

If the latter inequality is strict we have an immediate contradiction of \( x' \) solving e.m.p.;

if on the other hand \( p \ x' = p \ x'' > 0 \) then by continuity and strict monotonicity of \( u(\cdot) \) there exists \( \alpha \in (0, 1) \) close enough to 1 such that \( u(\alpha \ x'') > U' \) and \( p \ x' > p \ \alpha x'' \) which contradicts \( x' \) solving e.m.p..
(2.2): consider \( p' \) and \( p'' \) such that \( p''_l \geq p'_l \) but \( p''_k = p'_k \) for every \( k \neq l \).

Let \( x'' \) and \( x' \) be the solutions to the e.m.p. with \( p'' \) and \( p' \) respectively.

Then by definition of \( e(p, U) \)

\[
e(p'', U) = p'' x'' \geq p' x'' \geq p' x' = e(p', U).
\]

3. Homogeneous of degree 1 in \( p \).

**Proof:** The feasible set of the e.m.p. does not change when prices are multiplied by the factor \( k > 0 \).

Hence \( \forall k > 0 \), minimizing \( (k \ p) \ x \) on this set leads to the same answer. Let \( x^* \) be the solution, then:

\[
e(k \ p, U) = (k \ p) \ x^* = k \ e(p, U).
\]

**Proof:** let $p'' = t \ p + (1 - t) \ p'$ for $t \in [0, 1]$. Let $x''$ be the solution to e.m.p. for $p''$. Then

$$e(p'', U) = p'' \ x'' = t \ p \ x'' + (1 - t) \ p' \ x''$$

$$\geq t \ e(p, U) + (1 - t) \ e(p', U)$$

since $u(x'') \geq U$ and by definition of $e(p, U)$. ■

**Properties of the Hicksian demand functions:**

$$h(p, U)$$


$$\frac{\partial e(p, U)}{\partial p_l} = h_l(p, U)$$

**Proof:** by constrained envelope theorem. ■
2. Homogeneity of degree 0 in \( p \).

**Proof:** by Shephard’s lemma and the fact that the following theorem.

**Theorem.** If a function \( F(x) \) is homogeneous of degree \( r \) in \( x \) then \( (\partial F/\partial x_l) \) is homogeneous of degree \( (r - 1) \) in \( x \) for every \( l = 1, \ldots, L \).

**Proof:** Differentiating the identity that defines homogeneity of degree \( r \):

\[
F(k \ x) = k^r \ F(x) \quad \forall k > 0
\]

with respect to \( x_l \) we obtain:

\[
k \frac{\partial F(k \ x)}{\partial x_l} = k^r \frac{\partial F(x)}{\partial x_l}
\]
The latter equation corresponds to the definition of homogeneity of degree \((r - 1)\):

\[
\frac{\partial F(kx)}{\partial x_l} = k^{(r-1)} \frac{\partial F(x)}{\partial x_l}.
\]

**Euler Theorem.** If a function \(F(x)\) is homogeneous of degree \(r\) in \(x\) then:

\[
r F(x) = \nabla F(x) \cdot x
\]

**Proof:** Differentiating with respect to \(k\) the identity:

\[
F(kx) = k^r F(x) \quad \forall k > 0
\]
we obtain:

$$\nabla F(kx) \cdot x = rk^{(r-1)} F(x)$$

for $k = 1$ we obtain:

$$\nabla F(x) \cdot x = r F(x). \blacksquare$$

3. The matrix of cross-partial derivatives (Substitution matrix) with respect to $p$

$$S = \begin{pmatrix}
\frac{\partial h_1}{\partial p_1} & \cdots & \frac{\partial h_1}{\partial p_L} \\
\vdots & \ddots & \vdots \\
\frac{\partial h_L}{\partial p_1} & \cdots & \frac{\partial h_L}{\partial p_L}
\end{pmatrix}$$

is negative semi-definite and symmetric. (Main diagonal non-positive).
Proof: Symmetry follows from Shephard’s lemma and Young Theorem.

Indeed:

\[ \frac{\partial h_l}{\partial p_i} = \frac{\partial}{\partial p_i} \left( \frac{\partial e(p, U)}{\partial p_l} \right) = \frac{\partial}{\partial p_l} \left( \frac{\partial e(p, U)}{\partial p_i} \right) = \frac{\partial h_i}{\partial p_l} \]

While negative semi-definiteness follows from the concavity of \( e(p, U) \) and the observation that \( S \) is the Hessian of the function \( e(p, U) \). \( \blacksquare \)
Identities:

\[ V[p, e(p, U)] \equiv U \]
\[ x_l[p, e(p, U)] \equiv h_l(p, U) \quad \forall l \]
\[ e[p, V(p, m)] \equiv m \]
\[ h_l[p, V(p, m)] \equiv x_l(p, m) \quad \forall l \]

Slutsky decomposition:

start from the identity

\[ h_l(p, U) \equiv x_l[p, e(p, U)] \]

if the price \( p_i \) changes the effect is:

\[ \frac{\partial h_l}{\partial p_i} = \frac{\partial x_l}{\partial p_i} + \frac{\partial x_l}{\partial m} \frac{\partial e}{\partial p_i} \]
Notice that by Shephard’s lemma:

\[
\frac{\partial e}{\partial p_i} = h_i(p, U) = x_i[p, e(p, U)]
\]

then

\[
\frac{\partial h_l}{\partial p_i} = \frac{\partial x_l}{\partial p_i} + \frac{\partial x_l}{\partial m} x_i.
\]

or

\[
\frac{\partial x_l}{\partial p_i} = \frac{\partial h_l}{\partial p_i} - \frac{\partial x_l}{\partial m} x_i.
\]

Own price effect gives *Slutsky equation*:

\[
\frac{\partial x_l}{\partial p_l} = \frac{\partial h_l}{\partial p_l} - \frac{\partial x_l}{\partial m} x_l.
\]
Slutsky decomposition:

\[
\frac{\partial x_l}{\partial p_i} = \frac{\partial h_l}{\partial p_i} - \frac{\partial x_l}{\partial m} x_i.
\]

Slutsky equation:

\[
\frac{\partial x_l}{\partial p_l} = \frac{\partial h_l}{\partial p_l} - \frac{\partial x_l}{\partial m} x_l.
\]

This latter equation corresponds to the distinction between substitution and income effect:
*Substitution effect:*

\[ \frac{\partial h_l}{\partial p_l} \]

*Income effect:*

\[ \frac{\partial x_l}{\partial m} x_l \]
We know the sign of the *substitution effect* it is non-positive.

The sign of the income effect depends on whether the good is normal or inferior.

In the case that:

\[
\frac{\partial x_l}{\partial p_l} > 0
\]

we conclude that the good is *Giffen*.

This is not a realistic feature, inferior good with a big income effect.