Methods of Economic Investigation II (EC403)
Solutions to Problem Set #5

October 25, 2002

Solutions

1. (a) The process u_t is a Moving Average process of order 1. The autocorrelation function of u_t is:

$$\begin{aligned}\rho(1) &= \frac{\gamma(1)}{\gamma(0)} = \frac{\operatorname{cov}(u_t, u_{t-1})}{\operatorname{var}(u_t)} \\ \rho(s) &= 0 \ \forall s \geq 2.\end{aligned}$$

We calculate first the variance:

$$\operatorname{var}(u_t) = \operatorname{var}(\varepsilon_t) + 0.25\operatorname{var}(\varepsilon_{t-1}) - 2 \times 0.5\operatorname{cov}(\varepsilon_t, \varepsilon_{t-1}) = 1.25$$

then the covariance:

$$\operatorname{cov}(u_t, u_{t-1}) = E(u_t u_{t-1}) = E\left[(\varepsilon_t - 0.5\varepsilon_{t-1})(\varepsilon_{t-1} - 0.5\varepsilon_{t-2})\right] = -0.5E(\varepsilon_{t-1}^2) = -0.5$$

using the independence between the ε_t's. We get

$$\begin{aligned}\rho(1) &= \frac{-0.5}{1.25} = -0.4 \\ \rho(s) &= 0 \ \forall s \geq 2.\end{aligned}$$

(b) The process generating u_t is (weakly) stationary, its mean, variance, and autocovariance do not depend on t. The process generating y_t is not stationary because it is composed by deterministic trend. We can anyway calculate it's mean and variance to prove it:

$$\begin{aligned}E(y_t) &= \beta t \\ \operatorname{var}(y_t) &= 1.25\end{aligned}$$

We say that y_t is trend stationary because $y_t - \beta t$ is stationary.

(c) To construct the confidence interval of $\hat{\beta}$ we need its variance and mean:

$$\hat{\beta} = \frac{\sum_{t=1}^{T} t y_t}{\sum_{t=1}^{T} t^2} = \beta + \frac{\sum_{t=1}^{T} t u_t}{\sum_{t=1}^{T} t^2}$$

and $\hat{\beta}$ is an unbiased estimator of β. The variance is given by

$$\text{var}(\hat{\beta}) = \frac{1}{\left(\sum_{t=1}^{T} t^2\right)^2} \text{var}\left(\sum_{t=1}^{T} t u_t\right) = \frac{1}{\left(\sum_{t=1}^{T} t^2\right)^2} E\left[\left(\sum_{t=1}^{T} t u_t\right)^2\right],$$

where

$$\begin{aligned} E\left[\left(\sum_{t=1}^{T} t u_t\right)^2\right] &= E\left[\sum_{t=1}^{T}\sum_{s=1}^{T} t s u_t u_s\right] = E\left[\sum_{t=1}^{T} t^2 u_t^2\right] + E\left[t \neq s \sum\sum t s u_t u_s\right] \\ &= \sum_{t=1}^{T} t^2 E(u_t^2) + 2\sum_{t=1}^{T} t(t-1) E\left[u_t u_{t-1}\right] \\ &= 1.25 \sum_{t=1}^{T} t^2 + 2(-0.5)\sum_{t=1}^{T} t(t-1) \\ &= 1.25 \sum_{t=1}^{T} t^2 - \sum_{t=1}^{T} t^2 + \sum_{t=1}^{T} t = 0.25 \sum_{t=1}^{T} t^2 + \sum_{t=1}^{T} t. \end{aligned}$$

Therefore,

$$\text{var}(\hat{\beta}) = \frac{0.25 \sum_{t=1}^{T} t^2 + \sum_{t=1}^{T} t}{\left(\sum_{t=1}^{T} t^2\right)^2}$$

and the confidence interval is hence

$$\hat{\beta} \pm z_{\alpha/2}\sqrt{\text{var}(\hat{\beta})}.$$

Assuming $\varepsilon_t \sim i.i.d.\ N(0,1)$ this is exact, otherwise it is approximate.

2. (a) x_t is an AR(1) with drift, its mean and variance are hence:

$$\begin{aligned} E(x_t) &= \frac{\alpha}{1-\rho} = E(x_{t-1}) = \ldots \\ \text{var}(x_t) &= \frac{s^2}{1-\rho^2} = \text{var}(x_{t-1}) = \ldots \end{aligned}$$

Hence the autocovariance function is

$$\gamma(1) = \text{cov}(x_t, x_{t-1}) = E(x_t x_{t-1}) - E(x_t)E(x_{t-1}) = E\left[(\alpha + \rho x_{t-1} + u_t)(x_{t-1})\right] - \frac{\alpha^2}{(1-\rho)^2} =$$

$$= E\left[\alpha x_{t-1} + \rho x_{t-1}^2 + u_t x_{t-1}\right] - \frac{\alpha^2}{(1-\rho)^2} = \frac{\alpha^2}{1-\rho} + \rho E\left(x_{t-1}^2\right) - \frac{\alpha^2}{(1-\rho)^2} =$$

$$= \frac{\alpha^2}{1-\rho} + \frac{s^2}{1-\rho^2} + \frac{\alpha^2}{1-\rho^2} - \frac{\alpha^2}{(1-\rho)^2} = \rho\frac{s^2}{1-\rho^2} = \rho\gamma(0).$$

Generalizing,

$$\gamma(j) = \rho^j\gamma(0) = \rho^j\frac{s^2}{1-\rho^2}.$$

(b) The point estimate of x_{t+r} is obtained using the best linear predictor; suppose to know $\{x_t, x_{t-1}, \ldots\}$ and the actual value of the parameters, therefore

$$\hat{x}_{t+1|t} = E\left[x_{t+1} \mid x_t, x_{t-1}, \ldots\right] = \alpha + \rho x_t,$$

which is the one period ahead estimate of x_{t+1}. We can proceed in the same fashion for the two periods ahead:

$$\hat{x}_{t+2|t} = E\left[x_{t+2} \mid x_t, x_{t-1}, \ldots\right] = E\left[\alpha + \rho x_{t+1} + u_{t+2} \mid x_t, x_{t-1}, \ldots\right]$$

$$= \alpha + \rho(\alpha + \rho x_t) = \alpha + \alpha\rho + \rho^2 x_t.$$

Repeating this process for the r-period forward given the sample information available at time t, I_t, we have

$$\hat{x}_{t+r|t} = \alpha\sum_{j=0}^{r-1}\rho^j + \rho^r x_t,$$

which is, as already said, the best linear predictor of $x_{t+r|t}$. To calculate the interval forecast we need the variance of this forecast which is obtained from the one period ahead forecast error

$$e_{t+1} = x_{t+1} - \hat{x}_{t+1|t} = x_{t+1} - \alpha - \rho x_t = u_{t+1}.$$

The 2-periods ahead forecast error is

$$e_{t+2} = x_{t+2} - \hat{x}_{t+2|t} = x_{t+2} - \left[\alpha(1+\rho) + \rho^2 x_t\right] = u_{t+2} + \rho u_{t+1}$$

after substituting for x_{t+2}. In general we have

$$e_{t+r} = u_{t+r} + \rho u_{t+r-1} + \rho^2 u_{t+r-2} + \ldots + \rho^r u_{t+1} = \sum_{j=1}^{r}\rho^{r-j}u_{t+j}.$$

The forecast variance is thus

$$\operatorname{var}(e_{t+r} \mid I_t) = \sum_{j=1}^{r}\rho^{2(r-j)}\operatorname{var}(u_{t+j}) = s^2\sum_{j=1}^{r}\rho^{2(r-j)}$$

under the assumption we have about the errors. Hence the interval forecast is

$$\left(\hat{x}_{t+r|t-1}\right) \pm z_{\alpha/2}\sqrt{s^2\sum_{j=1}^{r}\rho^{2(r-j)}}$$

(c) We can do the same for y_t.

$$y_t = \gamma y_{t-1} + \beta E(x_{t+1} \mid I_t) + \varepsilon_t$$

and substituting for $E(x_{t+1} \mid I_t)$

$$y_t = \alpha\beta + \gamma y_{t-1} + \beta x_t + \varepsilon_t$$

then

$$\hat{y}_{t+1|t} = E(y_{t+1} \mid I_t) = \alpha\beta + \gamma y_t + \beta\rho E(x_{t+1} \mid I_t) = \alpha\beta + \gamma y_t + \beta\rho(\alpha + \rho x_t)$$

the associated error is

$$e_{t+1|t} = y_{t+1} - \hat{y}_{t+1|t} = \varepsilon_{t+1} + \beta\rho\left[x_{t+1} - \hat{x}_{t+1|t}\right] = \varepsilon_{t+1} + \beta\rho(u_{t+1})$$

hence the variance of the error forecast is

$$\mathrm{var}(e_{t+1} \mid I_t) = \sigma^2 + \beta^2\rho^2 s^2$$

then the interval forecast is

$$\hat{y}_{t+1|t} \pm z_{\alpha/2}\sqrt{\sigma^2 + \beta^2\rho^2 s^2}$$

(d) The estimation of the parameters proceed as follows:

- from the Ar(1) process

  $$x_{t+1} = \alpha + \rho x_t + u_{t+1}$$

  we get $\hat{\alpha}$ and $\hat{\rho}$ by OLS

- from

  $$y_t = \theta_0 + \theta_1 y_{t-1} + \theta_2 x_t + \varepsilon_t$$

  we get $\hat{\theta}_0$, $\hat{\theta}_1$, $\hat{\theta}_2$.

- Then using the relationship given by

  $$y_t = \alpha\beta + \gamma y_{t-1} + \beta\rho x_t + \varepsilon_t$$

  we have that

  $$\begin{aligned} \hat{\beta} &= \frac{\hat{\theta}_0}{\hat{\alpha}} \\ \hat{\gamma} &= \hat{\theta}_1 \\ \hat{\beta} &= \frac{\hat{\theta}_2}{\hat{\rho}} \end{aligned}$$

  Note that there's an alternative estimate of β.