Economics 481
Final Examination

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Instructions. Answer three questions. Explain all your answers.

1. Explain what is meant by a uniform law of large numbers. Explain the importance of uniform laws of large numbers in establishing consistency of the class of nonlinear estimators we have considered. Illustrate this with an example showing why pointwise convergence is not enough. Prove that the empirical distribution function based on an i.i.d. sample is uniformly consistent [you may assume that the population c.d.f. is continuous and strictly monotonic]. [20]

2. Suppose that you would like to know the dimensions of a table, the width (W), and the length (L), but that you only can get observations on the area $A = WL$. In fact, you have two separate research assistants who measure each quantity many times, but unfortunately they multiply their measurements together before passing them onto you. Specifically, suppose that

$$L_i \sim \chi^2(L_0) \text{ and } W_i \sim \chi^2(W_0), \quad i = 1, \ldots, n,$$

where W_0 and L_0 are the true unknown parameters, and that you observe $A_1, \ldots, A_n$, where $A_i = W_i L_i$. Note that the density function of a chi-squared with s degrees of freedom is $f(x; s) = e^{-x/2} x^{s/2-1} (1/2)^{s/2} (1/\Gamma(s/2))$ for any $s > 0$ and $x \geq 0$, where Γ is the gamma function, and has mean s and variance $2s$.

   (a) Show that if f_L is the density of L_i and f_W is the density of W_i, then

$$f_A(x; L, W) = \int_0^\infty f_L(\frac{x}{z}; L) \frac{1}{z} f_W(z; W) dz,$$

   where f_A is the density function for the variable A. [4]

(b) Because f_A is so complicated, it is not feasible to compute it analytically. Explain how you would obtain an unbiased simulation estimator of both $f_A(A_i;\theta)$ and $\partial f_A(A_i;\theta)/\partial\theta$, where $\theta = (L, W)$ [you should draw the same random variables for all values of θ]. Also show that it is not possible to get an unbiased simulator of the logarithmic derivative of the likelihood function $\partial l_n/\partial\theta$ of the observed sample $A_1, \ldots, A_n$, where $l_n(\theta) = \sum_{i=1}^n \log f_A(A_i;\theta)$. Thus explain now why it is not possible to obtain consistent estimators of θ_0 this way unless the number of simulation draws increases to infinity. [10]

(c) Show instead how to estimate θ_0 by the Generalized Method of Moments. [6]

3. Suppose you are interested in estimating a parameter vector $\theta = (\theta_1, \theta_2)$ from a sample moment condition $G_n(\theta)$ [for which $G(\theta) = EG_n(\theta)$ is equal to zero if and only if $\theta = \theta_0$], and that a preliminary estimator $\widehat{\theta}_2$ of θ_2 is available. Outline the conditions that are needed for the estimator $\widehat{\theta}_1$ that minimizes $\left\| G_n(\theta_1, \widehat{\theta}_2) \right\|$ to be consistent and asymptotically normal. Discuss the circumstances under which the distribution of $\widehat{\theta}_2$ affects the distribution of $\widehat{\theta}_1$.[20]

4. Suppose that $X_1, \ldots, X_n$ are i.i.d. and the following population moment conditions are known:

$$E\left\{ \begin{pmatrix} m_1(X_i, \theta_0) \\ m_2(X_i, \theta_0) \end{pmatrix} \right\} = 0,$$

for some unique θ_0, where $m_1(X_i, \theta) = X_i - \theta$ and $m_2(X_i, \theta) = (X_i - \theta)^3$.

(a) Define the class of Generalized Method of Moments estimators of θ_0 based on the above random sample of size n. [3]

(b) Give the asymptotic distribution of any member of this class and define the most efficient estimator. You should give the explicit formula [in terms of the population variance σ^2] in the case where X_i is normally distributed. [9]

(c) Show that the efficient GMM estimator is equivalent to solving the single equation

$$\alpha \frac{1}{n} \sum_{i=1}^n (X_i - \theta) + \beta \frac{1}{n} \sum_{i=1}^n (X_i - \theta)^3 = 0,$$

for some α, β that depend on θ_0. In the normal case, give α, β. Is this consistent with the Cramer-Rao theorem? [6]
[You may use the facts that for the normal distribution: $\kappa_4 = E(X_i - EX_i)^4 - 3\text{var}^2(X_i) = 0$ and $E(X_i - EX_i)^6 - 15\text{var}^3(X_i) = 0$].

5. Suppose that we are interested in estimating the α^{th} quantile of the continuous random variable y, i.e., the number $\theta_0(\alpha)$ such that $\Pr[y \leq \theta_0(\alpha)] = \alpha$, given a random sample $y_1, \ldots, y_n$ from this population. Define for any α with $0 \leq \alpha \leq 1$

$$H_\alpha(t) = \text{sign}(t) + (2\alpha - 1),$$

and consider the estimator $\widehat{\theta}$ that sets the following first order condition equal to zero

$$G_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} H_\alpha(y_i - \theta). \quad (1)$$

   (a) Prove that $\widehat{\theta}$ is consistent and derive its asymptotic distribution. [14]

   (b) Now suppose that y_i in (1) is actually a residual, i.e., $y_i = x_i - \beta z_i$, where x_i and z_i are observed variables, but β is an unknown slope parameter. Discuss the joint estimation of θ and β. [6]

6. Suppose that for some known α with $0 < \alpha < 1$ we have

$$\Pr(y_t = 1) = F(\beta_0 t^\alpha), \ t = 1, \ldots, n,$$

for some unknown parameter $\beta_0 > 0$, where F is the standard Cauchy c.d.f., with $F(x) = \int_{-\infty}^{x} f(x)dx$ and $f(x) = 1/\pi(1 + x^2)$.

   (a) Let $\widehat{\beta}$ be any zero of

$$G_n(\beta) = \frac{1}{n} \sum_{t=1}^{n} (y_t - F(\beta t^\alpha))t^\alpha$$

   over the compact set $B = [\epsilon, 1/\epsilon]$ for some positive ϵ such that $\beta_0 \in B$. Show that $\widehat{\beta}$ is consistent. You may use the fact that $F(x) = 1 - 1/x + O(1/x^3)$ as $x \to \infty$. [6]

   (b) Show that

$$n^{(1-\alpha)/2}(\widehat{\beta} - \beta_0) \to_d N(0, \frac{\beta_0^3}{(\alpha + 1)})$$

   [10]

   (c) If $\alpha = 1$, $\widehat{\beta}$ is inconsistent. Explain. [4]