Economics 481
Mock Exam

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Answer Three Questions. Show all your working.

1. State and prove a theorem that gives consistency for a GMM estimator without uniform convergence conditions. Discuss some applications of this result.

2. Suppose that y_i are i.i.d. and that

$$y_i = \theta + \sigma_i \varepsilon_i$$

$$\sigma_i^2 = \theta^2,$$

where $E(\varepsilon_i) = 0$ and $\text{var}(\varepsilon_i) = 1$.

   (a) Write down two vectors of population moment conditions that define θ and give the sample equivalents that can be used for estimating θ.

   (b) Compare the resulting estimator with the Maximum Likelihood Estimator based on the further assumption that $\varepsilon_i \sim N(0, 1)$.

   (c) Now describe how you might implement the maximum likelihood estimator by a two-step method with initial estimate $\widehat{\theta} = \overline{y}$. Will this two-step method be asymptotically efficient when the normality assumption is true?

3. Suppose that

$$y_i = m(z_i, \alpha) + \beta' w_i + \varepsilon_i,$$

where (y_i, z_i, w_i) are i.i.d. and $E(\varepsilon_i | z_i, w_i) = 0$. Here,

$$m(z, \theta) = \int_A h(x; z, \theta) dx,$$

where $A \subseteq \mathbb{R}$ is some given set.

(a) Describe how to compute $m(z, \theta)$ by simulation methods for each z, θ.

(b) Describe how you would estimate the parameter $\theta = (\alpha, \beta)$ from the dataset $\{(y_i, z_i, w_i),\ i = 1, \ldots, n\}$.

(c) Derive the asymptotic distribution of your estimator.

4. Let Z_i be i.i.d., and let $m(Z_i, \theta) = (m_1(Z_i, \theta), \ldots, m_q(Z_i, \theta))'$ be a vector of moment conditions for the scalar parameter θ for which $Em_j(Z_i, \theta) = 0$ if and only if $\theta = \theta_0$. You may also suppose that the matrix $Em(Z_i, \theta_0)m(Z_i, \theta_0)'$ is diagonal.

   (a) Describe the efficient GMM estimator of the parameter θ.

   (b) Derive the asymptotic distribution of the inefficient estimators $\widehat{\theta}_j$ that solve the first order condition

$$\frac{1}{n}\sum_{i=1}^{n} m_j(Z_i, \widehat{\theta}_j) = 0.$$

   (c) Consider the estimator

$$\widehat{\theta}(w) = \sum_{j=1}^{q} w_j \widehat{\theta}_j$$

   for some weights $w = (w_1, \ldots, w_q)$. Show that by appropriate choice of w the estimator $\widehat{\theta}(w_{opt})$ is asymptotically efficient.

5. Suppose that

$$y_i = \frac{1}{1 + \beta i^{\alpha}} + \varepsilon_i,$$

   where ε_i are i.i.d. with mean zero and finite variance. Here, α is a known value, while β is an unknown parameter of interest.

   (a) For what values of α is the nonlinear least squares estimator (NLLSE) consistent?

   (b) Derive the asymptotic distribution of the NLLSE when the estimator is consistent.
   [Hint: You should divide into the three cases $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$ and treat each case separately]

6. Let y be some continuous random variable and suppose that it is known that y has a unique median μ_0, defined as the value of μ that satisfies $\Pr(y \leq \mu_0) = 0.5$. Now define the interquartile range σ_0 to be any value that sets

$$\Pr\left(y \in [\mu_0 - \sigma_0, \mu_0 + \sigma_0]\right) = 0.5.$$

Suppose we have an i.i.d. sample $y_1, \ldots, y_n$ and we want to estimate the interquartile range from this dataset and that the median μ_0 is known. Consider the estimator $\widehat{\sigma}$ that satisfies the equation $G_n(\widehat{\sigma}) = o_p(1)$, where

$$G_n(\sigma) = \frac{1}{n}\sum_{i=1}^{n}\left(\{y_i \in [\mu_0 - \sigma, \mu_0 + \sigma]\} - 0.5\right),$$

where $\{A\}$ is the indicator of the set A.

(a) Derive the expectation $G(\sigma)$ of $G_n(\sigma)$.

(b) Establish consistency of the estimator $\widehat{\sigma}$ [you may need to make additional assumptions to guarantee that the interquartile range is unique, for example].

(c) Derive the asymptotic distribution of $\widehat{\sigma}$.

(d) Now suppose that μ_0 is replaced by βx_i for some observed i.i.d. random variable x_i and unknown parameter β. Now discuss joint estimation of the parameters β, σ.