Advanced Econometric Theory (EC481)
Problem Set #1 Solutions

1. The properties of the OLS estimator are well known and can be found in many textbooks. One set of sufficient conditions for consistency are that the error variance is finite [and the error is independent of the covariates] and

$$\lambda_{\min}(X'X) \xrightarrow{p} \infty.$$

In the i.i.d. case, sufficient conditions are that

$$E[\|x_i\varepsilon_i\|] < \infty,$$

i.e., one does not require finite variance of the error terms, and

$$0 < E[\|x_i x_i'\|] < \infty.$$

These conditions are just about averages and do not care whether the distributions involved are continuous or discrete. This is not the case for the LAD estimator as we now see. Suppose initially that we forget about the covariate, so that the LAD estimator is any minimizer of

$$Q_n(\alpha) = \frac{1}{n}\sum_{i=1}^{n} |y_i - \alpha|\,,$$

where

$$y_i = \alpha + \varepsilon_i,$$

with $\varepsilon_i = \pm 1/2$ with equal probabilities 1/2. The sample first order condition is then

$$G_n(\alpha) = \frac{1}{n}\sum_{i=1}^{n} \text{sign}(y_i - \alpha),$$

which has expectation

$$\begin{aligned} G(\alpha) &= \Pr(y_i > \alpha) - \Pr(y_i \le \alpha) \\ &= \Pr(\varepsilon_i > \alpha - \alpha_0) - \Pr(\varepsilon_i \le \alpha - \alpha_0) \\ &= \begin{cases} -1 & \text{if } \alpha \in [\alpha_0 + 1/2, \infty) \\ 0 & \text{if } \alpha \in [\alpha_0 - 1/2, \alpha_0 + 1/2) \\ 1 & \text{if } \alpha \in (-\infty, \alpha_0 - 1/2). \end{cases} \end{aligned}$$

Therefore, the entire interval $[\alpha_0 - 1/2, \alpha_0 + 1/2)$ is a zero of $G(\alpha)$, and this violates the identification condition of the consistency theorem. In class we further discussed what happens to the sample median in this case. We showed that it converges with probability one to the random variable that is equal to $\alpha_0 - 1/2$ with probability one half and $\alpha_0 + 1/2$ also with probability one half. This is because the probability that you hit exactly 50/50 heads out of n tosses decreases to zero as n increases. You are with probability tending to one strictly above or strictly below the equal probability mark.

2. Consider the following regression model

$$y_i = \theta x_i + \varepsilon_i,$$

where (y_i, x_i) are i.i.d. and the error term satisfies $E(\varepsilon_i | x_i) = 0$. Consider the following sample moment condition

$$G_n(\theta) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \theta x_i)^3 x_i.$$

We write

$$\begin{aligned} G_n(\theta) &= \frac{1}{n}\sum_{i=1}^{n}(\varepsilon_i + (\theta_0 - \theta)x_i)^3 x_i \\ &= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^3 x_i + \frac{1}{n}\sum_{i=1}^{n}3\varepsilon_i^2(\theta_0 - \theta)x_i^2 \\ &\quad + \frac{1}{n}\sum_{i=1}^{n}3\varepsilon_i(\theta_0 - \theta)^2 x_i^3 + \frac{1}{n}\sum_{i=1}^{n}(\theta_0 - \theta)^3 x_i^4. \end{aligned}$$

Provided

$$E[|\varepsilon_i^3 x_i|],\ E[|\varepsilon_i^2 x_i^2|],\ E[|\varepsilon_i x_i^3|],\ E[|x_i^4|] < \infty, \tag{1}$$

we can apply a law of large numbers so that

$$G_n(\theta) \xrightarrow{p} G(\theta) = E(\varepsilon_i^3 x_i) + 3(\theta_0 - \theta)E[\varepsilon_i^2 x_i^2] + 3(\theta_0 - \theta)^2 E[\varepsilon_i x_i^3] + (\theta_0 - \theta)^3 E(x_i^4),$$

which is a cubic polynomial in $\delta = \theta_0 - \theta$. By our assumption that $E(\varepsilon_i | x_i) = 0$, we have $E[\varepsilon_i x_i^3] = E[E[\varepsilon_i | x_i] x_i^3] = 0$, so that the third term in $G(\theta)$ is identically zero. If also $E(\varepsilon_i^3 | x_i) = 0$, then the first term is identically zero and

$$G(\theta) = 3(\theta_0 - \theta)E[\varepsilon_i^2 x_i^2] + (\theta_0 - \theta)^3 E(x_i^4).$$

Clearly, $G(\theta_0) = 0$. Provided $E[\varepsilon_i^2 x_i^2], E(x_i^4) > 0$, θ_0 is the unique zero of this cubic polynomial. In the absence of the condition $E(\varepsilon_i^3 x_i) = 0$, the estimator is inconsistent because $G(\theta_0) = E(\varepsilon_i^3 x_i) \neq 0$.