A Closed-form Estimator for the GARCH(1,1)-Model

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Abstract

We propose a closed-form estimator for the linear GARCH(1,1) model. The estimator has the advantage over the often used quasi-maximum-likelihood estimator (QMLE) that it can be easily implemented, and does not require the use of any numerical optimisation procedures or the choice of initial values of the conditional variance process. We derive the asymptotic properties of the estimator, showing $T^{(\kappa-1)/\kappa}$-consistency for some $\kappa \in (1,2)$ when the 4th moment exists and $\sqrt{T}$-asymptotic normality when the 8th moment exists. We demonstrate that a finite number of Newton-Raphson iterations using our estimator as starting point will yield asymptotically the same distribution as the QMLE when the 4th moment exists. A simulation study confirms our theoretical results.

1 Introduction

The estimation of the Bollerslev (1986) GARCH model is often carried out using the quasi-maximum likelihood estimator (QMLE). The asymptotic properties of this estimator are by now well established (see Jeantheau, 1998, Lee and Hansen, 1994, Lumsdaine, 1996, Ling and McAleer, 2003), and its finite sample properties have been examined through Monte Carlo studies (Fiorentini, Calzolari and Panattoni, 1996; Lumsdaine, 1995). Recently, Giraitis and Robinson (2001) have proposed using the Whittle likelihood estimator, which exploits only the second order properties of the series, but does so in an optimal way. Unfortunately, the calculation of both these estimators requires the use of numerical optimization procedures since closed form expressions are not available. Therefore, the resulting estimator depends on the implementation, with different optimization techniques leading to potentially different estimators. This has been demonstrated in the studies by Brooks, Burke and Persand (2001) and McCullough and Renfro (1999) where different commercially available software packages were used to estimate GARCH models by QML. Both studies reported markedly different outputs across the various packages, reflecting the different initialization and algorithmic strategies employed.

We propose a simple estimator of the parameters in the GARCH(1,1) model based on the second order properties of the series like Giraitis and Robinson (2001). We derive an ARMA representation of the squared GARCH process, and use the implied autocovariance function to define closed form estimators of the parameters of the GARCH model. This strategy has already been used in the literature on estimating standard ARMA models, see Tuan (1979) and Galbraith and Zinde-Walsh (1994), and in ARFIMA models recently by Mayoral (2004). The ARMA representation of GARCH models was already noted by Bollerslev (1986) while Franq and Zakoïan (2000), Gouriéroux and Jasiak (2001, p. 130)

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proposed to utilize this to obtain estimators of the parameters. However, Gouriéroux and Jasiak (2001) did not develop any specific estimator, while the least squares estimator of Franq and Zakoïan (2000) require numerical optimization. Baillie and Chung (2001) took a somewhat similar approach using a minimum distance estimator based on the autocorrelation function of the squared GARCH process; again, numerical optimization is required to obtain the actual estimator.

Our estimator can readily be implemented without using numerical optimization methods. Furthermore, the estimator does not require one to choose (arbitrary) initial values for the conditional variance process. In that sense, the proposed estimator is more robust compared to the aforementioned estimators. On the other hand, in order for the estimator to be consistent and asymptotically normally distributed, relatively strong assumptions about the moments of the GARCH process have to made. While consistency and asymptotic normality of the QMLE can be shown under virtually no moment restrictions of the GARCH process, we require a 4th moment to obtain consistency and $T^{(\kappa-1)/\kappa}$-convergence towards a so-called α-stable distribution with index $\kappa \in (1,2)$, and 8th moment for $\sqrt{T}$-asymptotic normality.

The $\sqrt{T}$-asymptotic normality result when the 8th moment exists follows from standard central limit theorems since the asymptotic variance in this case is well defined. When the 8th moment does not exist, we are still able to show that a limiting distribution exists, but it does not have second moment and the convergence rate is slowed down. The latter result is based on recent results by Mikosch and Stărică (2000). By combining our estimator with a finite order Newton-Raphson procedure one can achieve $\sqrt{T}$-asymptotic normality, and even full efficiency under Gaussianity of the rescaled errors, based on just a little more than 4th moments.

2 An ARMA Representation of the GARCH(1,1) Model

We consider the GARCH(1,1) process given by

$$y_t = \sigma_t z_t \tag{1}$$
$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha y_{t-1}^2. \tag{2}$$

We assume that $E[z_t|\mathcal{F}_{t-1}] = 0$ and $E[z_t^2|\mathcal{F}_{t-1}] = 1$, where $\mathcal{F}_t$ is the σ-field generated by $\{z_t, z_{t-1}, \ldots\}$. We can write $x_t \equiv y_t^2$ as

$$x_t = \omega + \phi x_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1},$$

where $\varepsilon_t = x_t - \sigma_t^2$ is a martingale difference sequence with respect to $\mathcal{F}_t$, $\phi = \alpha + \beta > 0$ and $\theta = -\beta < 0$. From this expression, we see that x_t is an (heteroskedastic) ARMA(1,1) process with parameters ϕ and θ. We shall throughout assume that $\phi < 1$ implying that $E[x_t] < \infty$, c.f. Bollerslev (1988). We introduce the covariance function of the process,

$$\gamma(k) = E\left[(x_{t+k} - E[x_t])(x_t - E[x_t])\right].$$

Assuming that $\{x_t\}$ is stationary with 2nd moment, $\gamma(\cdot)$ is well defined. Using standard results, we then have that the autocorrelation function, $\rho(k) = \gamma(k)/\gamma(0)$, solves the following set of Yule-Walker equations,

$$\rho(k) = \phi\rho(k-1),\ k = 2, 3, \ldots, \tag{3}$$
$$\rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+\theta^2+2\phi\theta}, \tag{4}$$

c.f. Harvey (1993, Chapter 1, 4.13a,b). These equations were also derived in Bollerslev (1988) and He and Teräsvirta (1999). We can express (4) as a quadratic equation in θ,

$$\theta^2 + b\theta + 1 = 0, \quad b \equiv \frac{\phi^2 + 1 - 2\rho(1)\phi}{\phi - \rho(1)}.$$

Observe that b is only well defined if $\phi \neq \rho(1)$. It is easily checked that $\rho(1) \geq \phi$ with equality if and only if $\phi^2 = 1$ or $\theta = 0$. The first case is ruled by our assumption that $\phi < 1$, while in the following we assume $\beta > 0$.[1] Under this assumption, $b > 2$ is well defined, and a solution to the quadratic equation is given by

$$\theta = \frac{-b + \sqrt{b^2 - 4}}{2}. \tag{5}$$

There is a second root which is reciprocal to the one stated here; however this has $|\theta| = \beta > 1$ which is ruled out by $\phi < 1$.[2] Finally, we observe that

$$\omega = \sigma^2(1 - \phi), \quad \sigma^2 \equiv E(y_t^2). \tag{6}$$

This last expression is utilized by Engle and Sheppard (2001) to profile out ω in their estimation procedure.

The expressions (4), (5) and (6) can now be used to obtain estimators of the parameters α, β and ω. First, we can estimate ϕ by $\hat{\phi} = \hat{\rho}(2)/\hat{\rho}(1)$, where $\hat{\rho}(\cdot)$ is the sample autocorrelation function of x_t, $\hat{\rho}(k) = \hat{\gamma}(k)/\hat{\gamma}(0)$ with

$$\hat{\gamma}(k) = \frac{1}{T-k} \sum_{t=1}^{T-k} \left(x_{t+k} - \hat{\sigma}^2\right)\left(x_t - \hat{\sigma}^2\right), \tag{7}$$

$$\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^{T} x_t. \tag{8}$$

Substituting the estimator of ϕ into (5), we obtain an estimator of θ,

$$\hat{\theta} = \frac{-\hat{b} + \sqrt{\hat{b}^2 - 4}}{2}, \quad \hat{b} \equiv \frac{\hat{\phi}^2 + 1 - 2\hat{\rho}(1)\hat{\phi}}{\hat{\phi} - \hat{\rho}(1)},$$

assuming that $\hat{b} \geq 2$. This leads to the following estimators of $\lambda = (\alpha, \beta, \omega)^\top$:

$$\hat{\alpha} = \hat{\theta} + \hat{\phi}, \quad \hat{\beta} = -\hat{\theta}, \quad \hat{\omega} = \hat{\sigma}^2(1 - \hat{\phi}). \tag{9}$$

In practice, this method may lead to $\hat{\phi} < 0$ or $\hat{\phi} > 1$. To deal with this problem, the estimator can be Winsorized (censored) at zero and one or at ϵ and $1 - \epsilon$ for small positive ϵ.

Note that

$$\phi = \sum_{j=1}^{\infty} w_j \frac{\rho(j+1)}{\rho(j)} \tag{10}$$

for any w_j sequence with $\sum_{j=1}^{\infty} w_j = 1$ so that a more general class of estimators can be defined based on this relationship. It can be expected that for sufficiently general class of weights one can obtain the same efficiency as the Whittle estimator of Giraitis and Robinson (2001). We do not pursue this approach to achieving efficiency since a more standard approach is available, see below. Nevertheless, some smoothing of the ratio of autocorrelations based on (10) may be desirable in practice.

[1] If $\beta = 0$, $\rho(1) = \alpha$, and we estimate α by the 1st order sample correlation.

[2] This corresponds to the invertibility-condition of the ARMA(1,1)-model.

3 Asymptotic Properties of the Estimator

We derive the asymptotic properties of our estimator $\hat{\lambda} \equiv (\hat{\alpha}, \hat{\beta}, \hat{\omega})^{\top}$ under the following set of assumptions:

A.1 The error process $\{z_t\}$ is i.i.d. with $E[z_t] = 0$ and $E[z_t^2] = 1$.

A.2 The observed process $\{y_t\}$ is strictly stationary with $E[(\beta + \alpha z_t^2)^2] < 1$.

A.3 $E[(\beta + \alpha z_t^2)^4] < 1$.

Under (A.1), the moment condition in (A.2) is necessary and sufficient for the GARCH model to have a strictly stationary solution with a 4th moment. We then assume that this is the one we have observed. Furthermore, this solution will be β-mixing with geometrically decreasing mixing coefficients, c.f. Carrasco and Chen (2002, Corollary 6). Assumption (A.3) is a strengthening of (A.2) implying that also the 8th moment of y_t exists, c.f. Carrasco and Chen (2002, Corollary 6).

In some of the results stated below, the i.i.d. assumption in (A.1) can be weakened to $\{z_t\}$ being a stationary martingale difference sequence; this is more realistic, allowing for dependence in the rescaled errors. In (A.2) we assume that we have observed the stationary version of the process. This can be replaced by the assumption that such a version exists, but this requires additional technicalities, c.f. Lee and Hansen (1994).

Under (A.1)-(A.2), we prove consistency of the estimator and derive its asymptotic distribution (towards which it converges with rate slower than $T^{-1/2}$). If additionally (A.3) holds, the estimator is shown to be $\sqrt{T}$-asymptotically normally distributed.

We first state a lemma which shows that the basic building blocks of our estimator are consistent and give their asymptotic distribution. Under (A.2), $E[y_t^4] < \infty$ but the 8th moment does not necessarily exist. One can show for an ARMA processes with homoskedastic errors that the sample autocorrelation function is asymptotically normally distributed with only a 2nd moment of the errors. In our case however, the errors, ε_t, are heteroskedastic and to obtain asymptotic normality the 4th moment, $E[\varepsilon_t^4] < \infty$, seems to be needed; see for example Hannan and Heyde (1972) for results in both the homoskedastic and heteroskedastic cases. The 4th moment of ε_t translates into the 8th moment of y_t.

The estimator $\hat{\gamma}(k)$ has a well defined asymptotic distribution without requiring the 8th moment of the GARCH process to exist however, but the limit will not be Gaussian. This result has been established in Mikosch and Stărică (2000), see also Basrak, Davis and Mikosch (2002), Davis and Mikosch (1998). They show that $\hat{\gamma}(k)$ converges in distribution towards a so-called stable, regularly varying distribution with index $\kappa \in (1, 2)$, see Samorodnitsky and Taqqu (1994) and Resnick (1987) for an introduction. The index κ is shown to be the solution to $E\left[\left(\alpha z_t^2 + \beta\right)^{\kappa}\right] = 1$. The convergence takes place with rate $T a_T^{-1}$ where $a_T = T^{1/\alpha} l(T)$ and $l(T)$ is a slowly varying function. The sequence a_T satisfies that $TP\left(y_t^4 > a_T\right) \to 1$, such that the index κ is a measure of the tail thickness.

Lemma 1 *Under (A.1)-(A.2), the estimators* $\hat{R}(m) = (\hat{\rho}(k))_{k=1,\dots,m}$ *and* $\hat{\sigma}^2$ *in (7)-(8) satisfy* $\hat{R}(m) \xrightarrow{p} R(m) = (\rho(k))_{k=1,\dots,m}$, $m \geq 1$, *and* $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$. *Furthermore,*

$$\sqrt{T}\left(\hat{\sigma}^2 - \sigma^2\right) \xrightarrow{d} N\left(0, V_{\sigma^2}\right), \tag{11}$$

where $V_{\sigma^2} = \gamma(0) + 2\sum_{k=1}^{\infty} \gamma(k)$, *and*

$$T^{(\kappa-1)/\kappa}(\hat{R}(m) - R(m)) \xrightarrow{d} \gamma^{-1}(0) W(m), \tag{12}$$

for some $\kappa \in (1, 2)$, *where* $W(m) = (Z_i - \rho(i) Z_0)_{i=1,\dots,m}$, *and* $(Z_i)_{i=0,\dots,m}$ *has a* κ*-stable distribution. If additionally (A.3) holds, then*

$$\sqrt{T}\left[\begin{array}{c} \hat{\sigma}^2 - \sigma^2 \\ \hat{R}(m) - R(m) \end{array}\right] \xrightarrow{d} N\left(0, \left[\begin{array}{cc} V_{\sigma^2} & O_{1\times m} \\ O_{m\times 1} & V_{\rho}(m) \end{array}\right]\right), \tag{13}$$

where $V_\rho(m)$ *is the* $m \times m$*-matrix given by*

$$V_\rho(m) = \mathrm{var}(Y_t) + \sum_{k=1}^{\infty} \mathrm{cov}(Y_t, Y_{t+k})$$

with Y_t *is an* $m \times 1$ *vector with* $Y_{t,i} = \left(y_t^2 - \sigma^2\right)\left(y_{t+i}^2 - \sigma^2\right)$, $i = 1, \ldots, m$.

Mikosch and Stărică (2000) establish a weak convergence result without the 4th moment condition (A.2). But in this case, one does not have any consistency result since the Law of Large Numbers does not hold. The main problem with the above weak convergence result of $\hat{\gamma}(k)$ under (A.1)-(A.2) is that the limit distribution is not in explicit form.

The consistency results and the weak convergence results in (11) and (13) in the above lemma can be proven without the i.i.d. assumption on $\{z_t\}$ in (A.1) by using martingale limit theory. This can be done by utilizing the ARMA structure of $\{x_t\}$ and applying the results of Hannan and Heyde (1972). It is not clear whether the more general weak convergence result in (12) can be extended to a non-i.i.d. setting however.

Observe that our estimator of λ can be expressed in terms of $\hat{\sigma}^2$ and $\hat{R}(2)$. A simple application of the continuous mapping theorem therefore yields the following result:

Theorem 2 *Under (A.1)-(A.2), the estimator* $\hat{\lambda} = (\hat{\alpha}, \hat{\beta}, \hat{\omega})^\top$ *given in (9) is consistent,* $\hat{\lambda} \xrightarrow{p} \lambda$, *and*

$$T^{(\kappa-1)/\kappa}(\hat{\lambda} - \lambda) \xrightarrow{d} D\left(\sigma^2, W(2)\right), \tag{14}$$

where $W(2)$ *and* κ *are as in Lemma 1, and the function* D *is given below in equations (20)-(22). If additionally (A.3) holds, then*

$$\sqrt{T}(\hat{\lambda} - \lambda) \xrightarrow{d} N(0, V), \text{ where} \tag{15}$$

$$V = \frac{\partial D\left(\sigma^2, \rho(1), \rho(2)\right)}{\partial\left(\sigma^2, \rho(1), \rho(2)\right)} \begin{bmatrix} V_{\sigma^2} & O_{1\times 2} \\ O_{2\times 1} & V_\rho(2) \end{bmatrix} \left(\frac{\partial D\left(\sigma^2, \rho(1), \rho(2)\right)}{\partial\left(\sigma^2, \rho(1), \rho(2)\right)}\right)^\top. \tag{16}$$

An estimator of the covariance matrix V in (16) can be obtained by first estimating V_{σ^2} and $V_\rho(2)$ using HAC variance estimators, see Robinson and Velasco (1997), and then substituting $\hat{\sigma}^2$, $\hat{\rho}(1)$ and $\hat{\rho}(2)$ into $\partial D(\sigma^2, \rho(1), \rho(2))/\partial(\sigma^2, \rho(1), \rho(2))$. One can alternatively use the analytic expressions of $\rho(k)$ to obtain an estimator of V_{σ^2}.

4 Efficiency Issues

We here give a brief discussion how one may improve on the efficiency of the closed form estimator, both in terms of convergence rate and asymptotic variance. The basic idea is to perform a number of Newton-Raphson (NR) iterations using either the Whittle objective function or the Gaussian quasi-likelihood. Given closed form estimates, one may wish to proceed to the QMLE or the Whittle estimator,[3] using the initial estimates as a starting point in the numerical optimization; this may help reduce numerical problems since our preliminary estimates are consistent. Alternatively, one can perform a number of NR-iterations which do not necessitate the use of any numerical optimization procedure. We define the following sequence of NR-estimators,

$$\hat{\lambda}_{k+1}^{\mathrm{NR}} = \hat{\lambda}_k^{\mathrm{NR}} - H_T^{-1}(\hat{\lambda}_k^{\mathrm{NR}}) S_T(\hat{\lambda}_k^{\mathrm{NR}}), \quad k \geq 1,$$

[3]Observe that the Whittle estimator proposed in Giraitis and Robinson (2001) in fact is the quasi-likelihood of the process $\{x_t\}$ assuming that it is Gaussian.

with initial value being the closed form estimator, $\hat{\lambda}_1^{\mathrm{NR}} = \hat{\lambda}$, $S_T(\lambda) = \partial Q_T(\lambda)/\partial\lambda$, $H_T(\lambda) = \partial^2 Q_T(\lambda)/\left(\partial\lambda\partial\lambda^\top\right)$, and $Q_T(\lambda)$ is the criterion function.[4] In the case of the QMLE, $Q_T(\lambda)$ is the Gaussian likelihood when assuming that $z_t \sim$ i.i.d. $N(0,1)$, while the Whittle estimator has $Q_T(\lambda)$ as the discrete frequency form, c.f. Giraitis and Robinson (2001, pp 611-612). For the QMLE, the NR-estimator takes the form of a GLS type estimator: Defining $\hat{\sigma}_{k,t}^2 = \hat{\omega}_k^{\mathrm{NR}} + \hat{\beta}_k^{\mathrm{NR}}\hat{\sigma}_{k,t-1}^2 + \hat{\alpha}_k^{\mathrm{NR}} y_{t-1}^2$, and $X_{k,t-1} = \left(1, y_{t-1}^2, \hat{\sigma}_{k,t-1}^2\right)'$, one can show that

$$\hat{\lambda}_{k+1}^{\mathrm{NR}} = \left(\sum_{t=1}^{n} \hat{\sigma}_{k,t}^{-2} X_{k,t-1} X_{k,t-1}'\right)^{-1} \left(\sum_{t=1}^{n} \hat{\sigma}_{k,t}^{-2} X_{k,t-1} y_t^2\right). \tag{17}$$

In general, the NR-estimator will satisfy

$$||\hat{\lambda}_{k+1}^{\mathrm{NR}} - \hat{\lambda}^{\mathrm{OP}}|| = O_P(||\hat{\lambda}_1^{\mathrm{NR}} - \hat{\lambda}^{\mathrm{OP}}||^{2^k}), \tag{18}$$

where $\hat{\lambda}^{OP} = \arg\max Q_T(\lambda)$ is the actual M-estimator, c.f. Robinson (1988, Theorem 2). Under regularity conditions, the M-estimator will satisfy

$$\sqrt{T}(\hat{\lambda}^{\mathrm{OP}} - \lambda) \xrightarrow{d} N\left(0, H^{-1}\Sigma H^{-1}\right), \tag{19}$$

where $H = E[H_T(\lambda_0)]$ and $\Sigma = E\left[S_T(\lambda) S_T(\lambda)^\top\right]$, see e.g. Lee and Hansen (1994, Theorem 2) and Giraitis and Robinson (2001, Theorem 1). Combining (18) and (19), we obtain the following result.

Theorem 3 *Assume that (19) holds together with (A.1)-(A.2). Then, with κ as in Lemma 1, for $k \geq 2 + [\log(\frac{\kappa}{2(\kappa-1)})/\log(2)]$,*

$$\sqrt{T}(\hat{\lambda}_k^{\mathrm{NR}} - \lambda) \xrightarrow{d} N\left(0, H^{-1}\Sigma H^{-1}\right).$$

If additionally (A.3) is satisfied, the above result holds for $k \geq 1$.

5 A Simulation Study

We now examine the quality of our estimator in finite sample through a Monte Carlo study. In particular, we are interested in the behaviour of it when the 4th moment does not exist. We simulate the GARCH process given in (1)-(2) with $z_t \sim$ i.i.d. $N(0,1)$ for four different sets of parameter values. For each choice of parameter values, we simulated 5000 data sets with $T = 200$, 400, 800 and 1000 observations. For each data set, we obtained the QMLE using the Matlab GARCH Toolbox, our ARMA estimator, and the GLS estimator in (17). To estimate ϕ, we used (10) with $w_j = 1/3$, $j = 1,2,3$, and $= 0$, $j > 3$, and Winsorized at ϵ and $1-\epsilon$ with $\epsilon = 0.001$. The GLS estimator reported here is based on one iteration. For the two first sets of parameter values, $\lambda = (0.2, 0.15, 0.25)^\top$ and $\lambda = (0.2, 0.25, 0.35)^\top$ both the 4th and 8th moment exist, for the third, $\lambda = (0.2, 0.35, 0.45)^\top$ only the 4th moment is well defined, while for the fourth, $\lambda = (0.2, 0.10, 0.89)^\top$ neither the 4th nor the 8th moment exist, but the 2nd does.

The results in terms of bias, variance and mean squared error (MSE) are reported in Tables 1-4 for the four different sets of parameters. For these results, we have discarded a small number of data sets (8 in total) where the Matlab GARCH Toolbox stalled and did not terminate the optimization procedure. We have also set the ARMA estimator equal to zero whenever it returned negative estimates.

[4] We here assume that $H_T(\hat{\lambda}_k^{NR})$ is invertible. Robinson (1988) discusses these issues in details.

For the first set of parameters, the ARMA and GLS estimators are serious competitors to the QMLE. For small GARCH effects, the closed form estimators seem to represent a good alternative to the QMLE. For the remaining three sets of parameters, the QMLE has significantly better performance compared to the two other estimators as expected. As α and β increase the variances of the ARMA and GLS estimators in general increase; this is consistent with the theory which predicts that the convergence rate of the estimator deteriorates as $\alpha + \beta$ increases. However, contrary to the theory, it seems as if the ARMA estimator remains consistent even if the 4th moment is not well defined. The GLS estimator based on two iterations yields a large improvement in the precision relative to the initial ARMA estimator, except for the estimation of ω. The improvement might have been even more pronounced if we had done further iterations.

6 Concluding Remarks

The procedure easily extends to the case where there is a mean process, say $y_t = \beta^\top x_t + \sigma_t z_t$ for some covariates x_t, by applying the above to the residuals from some preliminary fitting of the mean. Some aspects of the procedure extend to various multivariate cases and to GARCH(p, q). Specifically, in the multivariate case the relationships (3) and (6) continue to be useful and can be used as in Engle and Sheppard (2001) to reduce the dimensionality of the optimization space.

One may also consider other GARCH models that can be represented as an ARMA-like model. For example, suppose that the conditional variance is given as $\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha y_{t-1}^2 1(y_{t-1}^2 > 0) + \delta y_{t-1}^2 1(y_{t-1}^2 \leq 0)$. Then for $x_t \equiv y_t^2$ we have $x_t = \omega + \phi x_{t-1} + u_t$,where u_t is a martingale difference sequence with respect to $\mathcal{F}_{t-2}$ and $\phi = \beta + \alpha m_2^+ + \delta m_2^-$,where $m_2^+ = E[z_t^2 1(z_t > 0)]$ and $m_2^- = E[z_t^2 1(z_t \leq 0)]$. Furthermore, $\sigma^2 = \omega/(1-\phi)$ so that we can identify ω and ϕ as before from the variance and the second order covariance. To proceed further one needs to make strong assumptions about the distribution of z_t, for example symmetry about zero in which case $\phi = \beta + (\alpha + \delta)/2$.

Our estimator also allows for a simple t-test for no GARCH effect since the asymptotic distribution derived above does not require $\alpha, \beta > 0$ in contrast to the QMLE where the Taylor expansion used requires this.

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A Proofs

Proof of Lemma 1. The consistency part is a simple application of the Law of Large Numbers for stationary and ergodic processes since $E[x_t] < \infty$ and $E[x_t x_{t+k}] < \infty$ under (A.1)-(A.2). The weak convergence result in (12) follows from Mikosch and Stărică (2000, Section 5).

The convergence results in (11) and (13) is proved using a standard central limit theorem for markov chains which are strongly mixing with geometric rate, cf. Meyn and Tweedie (1993, Theorem 17.0.1). It is easily seen that

$$E\left[\left(y_s^2-\sigma^2\right)\left(y_t^2-\sigma^2\right)\left(y_{t+k}^2-\sigma^2\right)\right]=0,$$

for any $s,t \geq 0$ and $k > 0$. Thus, the asymptotic covariance between $\hat{\sigma}^2$ and $\hat{R}(m)$ is zero. ■

Proof of Theorem 2. We have that $\hat{\lambda} = D(\hat{\sigma}^2, \hat{\rho}(1), \hat{\rho}(2))$, where $D = (D_\alpha, D_\beta, D_\omega)^\top$ is given by

$$D_\alpha\left(\sigma^2,\rho(1),\rho(2)\right) = T(\rho(1), F(\rho(1),\rho(2))) + F(\rho(1),\rho(2)), \quad (20)$$

$$D_\beta\left(\sigma^2,\rho(1),\rho(2)\right) = -F(\rho(1),\rho(2)), \quad (21)$$

$$D_\omega\left(\sigma^2,\rho(1),\rho(2)\right) = \sigma^2(1-F(\rho(1),\rho(2))), \quad (22)$$

with

$$T(\rho,\phi) = \frac{-2\rho\phi+1+\phi^2-\sqrt{4\rho^2\phi^2+4\rho\phi-4\rho\phi^3+1-2\phi^2+\phi^4-4\rho^2}}{2(\rho-\phi)},$$

and $F(\rho(1),\rho(2)) = \rho(2)/\rho(1)$. It is easily seen that D is a continuous mapping of σ^2, $\rho(1)$ and $\rho(2)$. The convergence result in (14) now follows from the continuous mapping theorem together with the fact that $\hat{\sigma}^2$ is $\sqrt{T}$-consistent while $\hat{\rho}(1)$ and $\hat{\rho}(2)$ converge at a slower rate. The convergence result stated in (15) follows from the standard result for differentiable transformations of asymptotically normally distributed variables. ■

1 Tables

Table 1: MSE of the QML and ARMA-Estimator for $\lambda = (0.2, 0.15, 0.25)$

	QMLE				ARMA Estimator				GLS Estimator			
	T				T				T			
	200	400	800	1000	200	400	800	1000	200	400	800	1000
	5.28	5.71	5.75	5.78	22.93	18.20	13.46	12.24	0.22	1.79	4.09	4.47
$\omega = 0.20$	14.94	13.42	12.21	11.93	6.63	8.19	9.53	9.70	41.10	29.19	18.39	17.66
$(\times 10^{-3})$	20.22	19.13	17.96	17.70	29.56	26.39	22.99	21.94	41.32	30.98	22.48	22.12
	3.58	3.54	3.41	3.43	6.55	6.18	5.45	5.28	4.24	3.89	3.61	3.62
$\alpha = 0.15$	10.65	8.21	6.94	6.64	5.53	5.57	5.72	5.66	10.16	8.21	6.94	6.60
$(\times 10^{-3})$	14.23	11.75	10.35	10.07	12.08	11.75	11.17	10.94	14.40	12.10	10.55	10.21
	13.17	10.91	11.03	10.87	26.98	13.44	2.80	1.20	23.23	17.82	13.36	12.47
$\beta = 0.25$	53.40	47.99	37.23	35.02	159.20	140.20	112.55	102.24	36.47	37.76	35.20	34.44
$(\times 10^{-3})$	66.56	58.90	48.26	45.89	186.18	153.63	115.34	103.44	59.70	55.58	48.56	46.91

Notes: The three elements in each cell are, from top to bottom: squared bias, variance, and MSE.

Table 2: MSE of the QML and ARMA-Estimator for $\lambda = (0.2, 0.25, 0.35)$

	QMLE				ARMA Estimator				GLS Estimator			
	T				T				T			
	200	400	800	1000	200	400	800	1000	200	400	800	1000
	0.47	0.18	0.05	0.03	2.58	0.40	0.02	0.00	8.27	1.64	0.11	0.05
$\omega = 0.20$	9.83	5.71	3.13	2.56	18.71	14.39	9.15	7.56	82.38	64.20	32.03	22.80
$(\times 10^{-3})$	10.29	5.90	3.18	2.59	21.29	14.79	9.17	7.57	90.65	65.84	32.14	22.85
	0.00	0.00	0.00	0.00	9.28	4.67	1.86	1.24	0.25	0.10	0.04	0.03
$\alpha = 0.25$	12.77	6.36	3.26	2.61	10.58	10.22	7.49	6.34	13.60	7.48	3.93	3.16
$(\times 10^{-3})$	12.78	6.36	3.26	2.61	19.86	14.89	9.34	7.57	13.86	7.58	3.97	3.19
	2.62	1.00	0.27	0.16	67.08	24.23	6.31	3.27	9.36	2.30	0.12	0.02
$\beta = 0.35$	51.53	32.24	18.16	14.87	90.51	84.25	58.06	48.39	59.68	44.97	29.17	24.53
$(\times 10^{-3})$	54.15	33.23	18.42	15.03	157.58	108.48	64.37	51.65	69.04	47.27	29.29	24.55

Notes: The three elements in each cell are, from top to bottom: squared bias, variance, and MSE.

Table 3: MSE of the QML and ARMA-Estimator for $\lambda = (0.2, 0.35, 0.45)$

	QMLE				ARMA Estimator				GLS Estimator			
	T				T				T			
	200	400	800	1000	200	400	800	1000	200	400	800	1000
	1.46	0.29	0.06	0.04	1.40	1.33	0.87	0.71	13.22	1.27	0.16	0.33
$\omega = 0.20$	13.37	5.56	2.41	1.81	53.69	33.83	21.97	18.98	418.89	742.68	83.66	73.06
$(\times 10^{-3})$	14.83	5.84	2.47	1.85	55.09	35.16	22.84	19.69	432.11	743.95	83.82	73.38
	0.00	0.00	0.00	0.00	24.96	14.83	8.90	7.52	0.28	0.01	0.00	0.01
$\alpha = 0.35$	15.14	7.54	3.72	2.91	16.16	15.17	12.62	12.05	19.40	11.25	6.36	5.22
$(\times 10^{-3})$	15.14	7.54	3.72	2.91	41.12	30.00	21.52	19.56	19.68	11.27	6.36	5.23
	2.54	0.46	0.09	0.06	24.05	9.89	4.77	3.98	4.58	0.06	0.38	0.50
$\beta = 0.45$	30.40	14.44	6.48	5.02	83.22	69.79	52.84	48.23	61.15	39.08	23.74	19.99
$(\times 10^{-3})$	33.53	14.90	6.57	5.08	107.28	79.68	57.62	52.21	65.74	39.14	24.12	20.48

Notes: The three elements in each cell are, from top to bottom: squared bias, variance, and MSE.

Table 4: MSE of the QML and ARMA-Estimator for $\lambda = (0.2, 0.10, 0.89)$

	QMLE				ARMA Estimator				GLS Estimator			
	T				T				T			
	200	400	800	1000	200	400	800	1000	200	400	800	1000
	1×10^3	96.83	10.65	5.80	4×10^3	3×10^3	1×10^3	1×10^3	2×10^4	2×10^3	224.26	99.02
$\omega = 0.20$	6×10^3	481.07	49.35	21.18	3×10^4	1×10^4	6×10^3	5×10^3	7×10^4	7×10^3	1×10^3	909.83
$(\times10^{-3})$	8×10^3	577.90	60.00	26.98	4×10^4	1×10^4	7×10^3	7×10^3	9×10^4	9×10^3	2×10^3	1×10^3
	0.01	0.00	0.00	0.00	0.05	0.15	0.27	0.32	0.01	0.07	0.09	0.12
$\alpha = 0.10$	3.39	1.20	0.54	0.42	3.15	3.17	3.13	3.07	7.81	5.42	3.54	3.01
$(\times10^{-3})$	3.40	1.20	0.54	0.42	3.19	3.32	3.39	3.38	7.83	5.49	3.63	3.13
	6.80	0.56	0.06	0.04	10.81	9.21	6.26	5.90	62.22	7.12	0.01	0.16
$\beta = 0.89$	29.35	4.42	0.92	0.53	45.34	35.49	22.91	20.16	86.40	38.41	13.55	10.12
$(\times10^{-3})$	36.15	4.98	0.98	0.56	56.15	44.69	29.17	26.06	148.62	45.53	13.56	10.27

Notes: The three elements in each cell are, from top to bottom: squared bias, variance, and MSE.