$$
\begin{aligned}
& [\varepsilon,\varepsilon]^{\overline{n}} \\
= & \sum_{i=1}^{\overline{n}-1}\left(\varepsilon_{t_{iK+1}}-\varepsilon_{t_{(i-1)K+1}}\right)^2 \\
= & \sum_{i=1}^{\overline{n}-1}\left(\mu_{t_{iK+1}}-\mu_{t_{(i-1)K+1}}+n^{-\alpha/2}\sigma_\epsilon\left(\frac{iK+1}{n}\right)\epsilon_{t_{iK+1}}-n^{-\alpha/2}\sigma_\epsilon\left(\frac{(i-1)K+1}{n}\right)\epsilon_{t_{(i-1)K+1}}\right)^2 && (1)\\
= & \sum_{i=1}^{\overline{n}-1}\left(\mu_i-\mu_{i-1}+n^{-\alpha/2}\sigma_i\epsilon_i-n^{-\alpha/2}\sigma_{i-1}\epsilon_{i-1}\right)^2,\ \text{i.e., change notation only} && (2)\\
= & [\mu,\mu]^{\overline{n}}+n^{-\alpha/2}\sum_{i=1}^{\overline{n}-1}\left(\mu_i-\mu_{i-1}\right)\left(\sigma_i\epsilon_i-\sigma_{i-1}\epsilon_{i-1}\right)+n^{-\alpha}\sum_{i=1}^{\overline{n}-1}\left(\sigma_i\epsilon_i-\sigma_{i-1}\epsilon_{i-1}\right)^2 && (3)\\
= & [\mu,\mu]^{\overline{n}}+A_1+A_2 && (4)\\
= & [\mu,\mu]^{\overline{n}}+n^{-\alpha/2}O_p\left(\frac{1}{\overline{n}^{1/2}}\right)+2\overline{n}n^{-\alpha}\sigma_\epsilon^2+O_p\left(n^{-\alpha}\overline{n}^{1/2}\right) && (5)\\
= & 2\overline{n}n^{-\alpha}\sigma_\epsilon^2+O_p\left(n^{-\alpha}\overline{n}^{1/2}\right)+O_p\left(n^{-\alpha/2}\overline{n}^{-1/2}\right),\ \text{last term } =o_p\left(n^{-\alpha}\overline{n}^{1/2}\right)\ \text{if } \alpha<2-2\beta && (6)
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= n^{-\alpha/2}\sum_{i=1}^{\overline{n}-1}\left(\mu_i-\mu_{i-1}\right)\left(\sigma_i\epsilon_i-\sigma_{i-1}\epsilon_{i-1}\right)\\
&= n^{-\alpha/2}O\left(\frac{1}{\overline{n}}\right)\sum_{i=1}^{\overline{n}-1}\sigma_i\epsilon_i\\
&= n^{-\alpha/2}O_p\left(\frac{1}{\overline{n}^{1/2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= n^{-\alpha}\sum_{i=1}^{\overline{n}-1}\left(\sigma_i\epsilon_i-\sigma_{i-1}\epsilon_{i-1}\right)^2\\
&= 2n^{-\alpha}\sum_{i=1}^{\overline{n}-1}\left(\sigma_i\epsilon_i\right)^2+O_p\left(n^{-\alpha}\right)-2n^{-\alpha}\sum_{i=1}^{\overline{n}-1}\sigma_i\sigma_{i-1}\epsilon_i\epsilon_{i-1}\\
&= 2n^{-\alpha}\sum_{i=1}^{\overline{n}-1}\left(\left(\sigma_i\epsilon_i\right)^2-\sigma_i^2\right)+2n^{-\alpha}\sum_{i=1}^{\overline{n}-1}\sigma_i^2+O_p\left(n^{-\alpha}\right)-2n^{-\alpha}\sum_{i=1}^{\overline{n}-1}\sigma_i\sigma_{i-1}\epsilon_i\epsilon_{i-1}\\
&= 2n^{-\alpha}O_p\left(\overline{n}^{1/2}\right)+2\overline{n}n^{-\alpha}\sigma_\epsilon^2+O_p\left(n^{-\alpha}\overline{n}^{1/2}\right)\\
&= 2\overline{n}n^{-\alpha}\sigma_\epsilon^2+O_p\left(n^{-\alpha}\overline{n}^{1/2}\right)
\end{aligned}
$$

Cross term.

$$
\begin{aligned}
& [X,\varepsilon]^{\overline{n}} \\
= & \sum_{i=1}^{\overline{n}-1}\left(\varepsilon_{t_{iK+1}}-\varepsilon_{t_{(i-1)K+1}}\right)\left(X_{t_{iK+1}}-X_{t_{(i-1)K+1}}\right) \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{\overline{n}-1}\left(\varepsilon_{t_{iK+1}}-\varepsilon_{t_{(i-1)K+1}}\right)\left(u_{t_{iK+1}}+u_{t_{iK}}+u_{t_{iK-1}}+...+u_{t_{(i-1)K+2}}\right) \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{\overline{n}-1}\left(\mu_{t_{iK+1}}-\mu_{t_{(i-1)K+1}}\right)\left(u_{t_{iK+1}}+u_{t_{iK}}+u_{t_{iK-1}}+...+u_{t_{(i-1)K+2}}\right)+ \\
& +n^{-(1+\alpha)/2}\sum_{i=1}^{\overline{n}-1}\left(\sigma_\epsilon\left(\frac{iK+1}{n}\right)\epsilon_{t_{iK+1}}-\sigma_\epsilon\left(\tfrac{(i-1)K+1}{n}\right)\epsilon_{t_{(i-1)K+1}}\right)\left(u_{t_{iK+1}}+u_{t_{iK}}+u_{t_{iK-1}}+...+u_{t_{(i-1)K+2}}\right) \\
= & O_p\left(n^{-1/2}\overline{n}^{-1}\left(K\overline{n}\right)^{1/2}\right)+n^{-(1+\alpha)/2}\sum_{i=1}^{\overline{n}-1}\sigma_\epsilon\left(\frac{iK+1}{n}\right)\epsilon_{t_{iK+1}}u_{t_{iK+1}}+ \\
& +n^{-(1+\alpha)/2}\sum_{i=1}^{\overline{n}-1}\sigma_\epsilon\left(\frac{iK+1}{n}\right)\epsilon_{t_{iK+1}}\left(u_{t_{iK}}+u_{t_{iK-1}}+...+u_{t_{(i-1)K+2}}\right)- \\
& -n^{-(1+\alpha)/2}\sum_{i=1}^{\overline{n}-1}\sigma_\epsilon\left(\tfrac{(i-1)K+1}{n}\right)\epsilon_{t_{(i-1)K+1}}\left(u_{t_{iK+1}}+u_{t_{iK}}+u_{t_{iK-1}}+...+u_{t_{(i-1)K+2}}\right) \\
= & O_p\left(\overline{n}^{-1}\right)+C_1+C_2+C_3 \\
= & O_p\left(\overline{n}^{-1}\right)+n^{-(1+\alpha)/2}O_p\left(\overline{n}^{1/2}\right)+n^{-(1+\alpha)/2}\overline{n}\rho_{u\varepsilon}\sigma_\epsilon+n^{-(1+\alpha)/2}O_p\left(n^{1/2}\right)+n^{-(1+\alpha)/2}O_p\left(n^{1/2}\right) \\
= & n^{-(1+\alpha)/2}\overline{n}\rho_{u\varepsilon}\sigma_\epsilon+O_p\left(n^{-\alpha/2}\right)
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= n^{-(1+\alpha)/2}\sum_{i=1}^{\overline{n}-1}\sigma_\epsilon\left(\frac{iK+1}{n}\right)\epsilon_{t_{iK+1}}u_{t_{iK+1}} \\
&= n^{-(1+\alpha)/2}\sum_{i=1}^{\overline{n}-1}\left(\sigma_\epsilon\left(\frac{iK+1}{n}\right)\epsilon_{t_{iK+1}}u_{t_{iK+1}}-\sigma_\epsilon\left(\frac{iK+1}{n}\right)\rho_{u\varepsilon}\right)+\rho_{u\varepsilon}n^{-(1+\alpha)/2}\sum_{i=1}^{\overline{n}-1}\sigma_\epsilon\left(\frac{iK+1}{n}\right) \\
&= n^{-(1+\alpha)/2}O_p\left(\overline{n}^{1/2}\right)+n^{-(1+\alpha)/2}\overline{n}\rho_{u\varepsilon}\sigma_\epsilon
\end{aligned}
$$

$$
\begin{aligned}
C_2 &= n^{-(1+\alpha)/2}\sum_{i=1}^{\overline{n}-1}\sigma_\epsilon\left(\frac{iK+1}{n}\right)\epsilon_{t_{iK+1}}\left(u_{t_{iK}}+u_{t_{iK-1}}+...+u_{t_{(i-1)K+2}}\right) \\
&= n^{-(1+\alpha)/2}O_p\left(\left(K\overline{n}\right)^{1/2}\right)=n^{-(1+\alpha)/2}O_p\left(n^{1/2}\right)
\end{aligned}
$$

$$
\begin{aligned}
C_3 &= n^{-(1+\alpha)/2}\sum_{i=1}^{\overline{n}-1}\sigma_\epsilon\left(\tfrac{(i-1)K+1}{n}\right)\epsilon_{t_{(i-1)K+1}}\left(u_{t_{iK+1}}+u_{t_{iK}}+u_{t_{iK-1}}+...+u_{t_{(i-1)K+2}}\right) \\
&= n^{-(1+\alpha)/2}O_p\left(\sqrt{K\overline{n}}\right)=n^{-(1+\alpha)/2}O_p\left(n^{1/2}\right)
\end{aligned}
$$