# Weak instruments

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# IV regression with weak instruments

Bound, Jaeger, and Baker (1995) pointed out that the quarter of birth instruments explain only a tiny proportion of the variation in schooling. This leads to two distinct problems:

- The 2SLS estimator with weak instruments is biased in small samples.
- Any inconsisency from a small violation of the exclusion restriction gets magnified by weak instruments.

# The 2SLS bias

Start with the 2SLS small sample bias. To get an intuition for this situation, look at the simplest formulation of the IV problem:

$$\begin{aligned} y_i &= \beta x_i + \eta_i \qquad \text{(structural equation)} \\ x_i &= \pi_1 z_i + \xi_i \qquad \text{(first stage)} \end{aligned}$$

# The small sample behavior with one instrument

- If an instrument is basically irrelevant then $\pi_1 \approx 0$. Recall

$$\beta_{IV} = \frac{cov\left(y_i, z_i\right)}{cov\left(x_i, z_i\right)}$$

but

$$cov\left(x_i, z_i\right) = cov\left(\pi_1 z_i + \xi_i, z_i\right) = \pi_1 \sigma_Z^2$$

So if $\pi_1 = 0$,

$$cov\left(x_i, z_i\right) = 0$$

and the IV estimator doesn't exist.

- Even when $\pi_1$ is truely zero, in any finite sample the sample analogue to $cov\left(x_i, z_i\right)$ will not be exactly zero. But this is of little comfort as the sampling variation in $cov\left(x_i, z_i\right)$ is not helpful to estimate $\beta$.

# The 2SLS bias with many instruments

With multiple instruments the first stage is:

$$x = Z\pi + \xi.$$

OLS estimates are biased because $\eta_i$ is correlated with $\xi_i$. The instruments $Z_i$ are uncorrelated with $\xi_i$ by construction and uncorrelated with $\eta_i$ by assumption.
The 2SLS estimator is

$$\widehat{\beta}_{2SLS} = \left(x'P_Z x\right)^{-1} x'P_Z y = \beta + \left(x'P_Z x\right)^{-1} x'P_Z \eta$$

where $P_Z = Z(Z'Z)^{-1}Z'$ is the projection matrix that produces fitted values from a regression of $x$ on $Z$. Substitute the first stage for $x$ in $x'P_Z\eta$ to get

$$\begin{aligned}\widehat{\beta}_{2SLS} - \beta &= \left(x'P_Z x\right)^{-1} \left(\pi'Z' + \xi'\right) P_Z \eta \\ &= \left(x'P_Z x\right)^{-1} \pi'Z'\eta + \left(x'P_Z x\right)^{-1} \xi'P_Z \eta\end{aligned}$$

# Group asymptotics

The expectation of this expression is hard to evaluate because the expectation operator does not pass through the inverse $(x'P_Z x)^{-1}$, a nonlinear function.

- Trick: group asymptotics. Still use an asymptotic argument but let the number of instruments grow at the same rate as the sample size. This "keeps the instruments weak."
- Group asymptotics gives us something like an expectation, it essentially says that we can take these expectations through non-linear functions anyway:

$$E[\widehat{\beta}_{2SLS} - \beta] \approx \left(E[x'P_Z x]\right)^{-1} E[\pi' Z' \eta] + \left(E[x'P_Z x]\right)^{-1} E[\xi' P_Z \eta].$$

  This approximation is much better than the usual first-order asymptotic approximation invoked in large-sample theory, so it gives us a good measure of the finite-sample behavior of the 2SLS estimator.

# The 2SLS bias with many instruments

Remember the instruments $Z_i$ are uncorrelated with $\xi_i$ and $\eta_i$. Therefore $E[\pi' Z' \eta] = 0$, and we have

$$\begin{aligned} E[\widehat{\beta}_{2SLS} - \beta] &\approx \left(E[x' P_Z x]\right)^{-1} E[\pi' Z' \eta] + \left(E[x' P_Z x]\right)^{-1} E[\xi' P_Z \eta] \\ &= \left(E[x' P_Z x]\right)^{-1} E[\xi' P_Z \eta]. \end{aligned}$$

Substitute in the first stage again.

$$E[\widehat{\beta}_{2SLS} - \beta] \approx \left(E[\left(\pi' Z' + \xi'\right) P_Z \left(Z\pi + \xi\right)]\right)^{-1} E[\xi' P_Z \eta].$$

Note that $E[\pi' Z' \xi] = 0$, so we get no cross-terms:

$$E[\widehat{\beta}_{2SLS} - \beta] \approx \left[E\left(\pi' Z' Z \pi\right) + E(\xi' P_Z \xi)\right]^{-1} E\left(\xi' P_Z \eta\right).$$

# The 2SLS bias with many instruments

Matrix algebra trick: $\xi' P_Z \xi$ is a scalar, therefore equal to its trace; the trace is a linear operator which passes through expectations and is invariant to cyclic permutations; finally, the trace of $P_Z$, an idempotent matrix, is equal to it's rank, $\mathrm{Q}$. Using these facts

$$
\begin{aligned}
E\left(\xi' P_Z \xi\right) &= E\left[\operatorname{tr}\left(\xi' P_Z \xi\right)\right] \\
&= E\left[\operatorname{tr}\left(P_Z \xi \xi'\right)\right] \\
&= \operatorname{tr}\left(P_Z E\left[\xi \xi'\right]\right) \\
&= \operatorname{tr}\left(P_Z \sigma_\xi^2 I\right) \\
&= \sigma_\xi^2 \operatorname{tr}\left(P_Z\right) \\
&= \sigma_\xi^2 \mathrm{Q},
\end{aligned}
$$

where we have assumed that $\xi_i$ is homoskedastic. Similarly, applying the trace trick to $\xi' P_Z \eta$ shows that this term is equal to $\sigma_{\eta\xi}\mathrm{Q}$.

# The 2SLS bias with many instruments

Now we have

$$E[\widehat{\beta}_{2SLS} - \beta] \approx \sigma_{\eta\xi} \mathrm{Q} \left[ E\left(\pi' Z' Z \pi\right) + \sigma_{\xi}^2 \mathrm{Q} \right]^{-1}$$
$$= \frac{\sigma_{\eta\xi}}{\sigma_{\xi}^2} \left[ \frac{E\left(\pi' Z' Z \pi\right) / \mathrm{Q}}{\sigma_{\xi}^2} + 1 \right]^{-1}.$$

Note that

$$F = \frac{E\left(\pi' Z' Z \pi\right) / \mathrm{Q}}{\sigma_{\xi}^2}$$

is the population $F$-statistic for the joint significance of all regressors in the first stage regression and hence

$$E[\widehat{\beta}_{2SLS} - \beta] \approx \frac{\sigma_{\eta\xi}}{\sigma_{\xi}^2} \frac{1}{F + 1}.$$

# The 2SLS bias with a zero first stage

Suppose the first stage coefficients $\pi$ are truely zero. Then $F = 0$. Furthermore

$$\sigma_x^2 = \sigma_\xi^2.$$

Hence

$$E[\widehat{\beta}_{2SLS} - \beta] \approx \frac{\sigma_{\eta\xi}}{\sigma_x^2}$$

but this is just the bias in the OLS estimator, since

$$\begin{aligned} \beta_{OLS} &= \frac{cov(y_i, x_i)}{var(x_i)} = \frac{cov(\beta x_i + \eta_i, x_i)}{var(x_i)} \\ &= \beta + \frac{cov(\eta_i, x_i)}{var(x_i)} = \frac{\sigma_{\eta\xi}}{\sigma_x^2} \end{aligned}$$

since $cov(\eta_i, x_i) = \sigma_{\eta\xi}$ if $\pi = 0$.

# 2SLS is biased towards OLS with weak instruments

Where does this come from?

- If $\pi$ is truely zero, then any variation in $\widehat{x}_i$ in the sample just comes from $\xi_i$. So, the variation in $\widehat{x}_i$ is no different from the variation in $x_i$, and hence OLS and 2SLS have to estimate the same quantity on average.
- If $\pi$ is not truely zero but $F$ is small, then 2SLS will be biased towards OLS.

# What does the weak instrument bias depend on?

The weak instrument bias tends to get worse as we add more (weak) instruments. To see this consider

$$F = \frac{E\left(\pi' Z' Z \pi\right) / \mathrm{Q}}{\sigma^2_{\xi}}$$

Suppose you have some existing instruments, and you add new ones with no additional exploratory power. I.e. the $\pi$ coefficients on the additional instruments will be zero.

- $\pi' Z' Z \pi$ will remain the same as before adding more instruments.
- Since the first stage regression is unchanged by the additional instruments, $\sigma^2_{\xi}$ will also remain the same.
- $\mathrm{Q}$ will go up.

As a result, $F$ will go down, and the 2SLS bias will get worse.

# Summing up on the bias

With weak instruments

- 2SLS is biased towards OLS.
- The bias will tend to be worse when there are many overidentifying restrictions (many instruments compared to endogenous regressors).
- Just identified IV is approximately unbiased (or less biased) even with weak instruments (although it is not possible to see this from the bias formula).
- Estimated standard errors of 2SLS and IV estimators may be too small.

# LIML

- There are alternative estimators, which have better small sample properties than 2SLS with weak instruments. One such estimator is *LIML* (limited information maximum likelihood).
- LIML is a linear combination of the OLS and 2SLS estimate (with the weights depending on the data), and the weights happen to be such that they (approximately) eliminate the 2SLS bias.

# A Monte Carlo Experiment

Simulate data from the following model

$$
\begin{aligned}
y_i &= \beta x_i + \eta_i \\
x_i &= \sum_{j=1}^{Q} \pi_j z_{ij} + \xi_i
\end{aligned}
$$

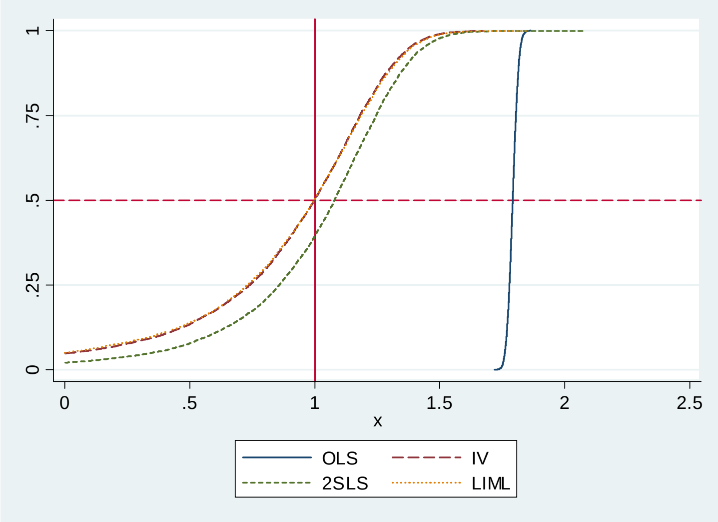
with $\beta = 1$, $\pi_1 = 0.1$, $\pi_j = 0$ for $j = 2, ..., Q$,

$$
\left( \begin{array}{c} \eta_i \\ \xi_i \end{array} \right) \Bigg| Z \sim N \left( \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{cc} 1 & 0.8 \\ 0.8 & 1 \end{array} \right) \right),
$$

where the $z_{ij}$ are independent, normally distributed random variables with mean zero and unit variance. The sample size is 1000.

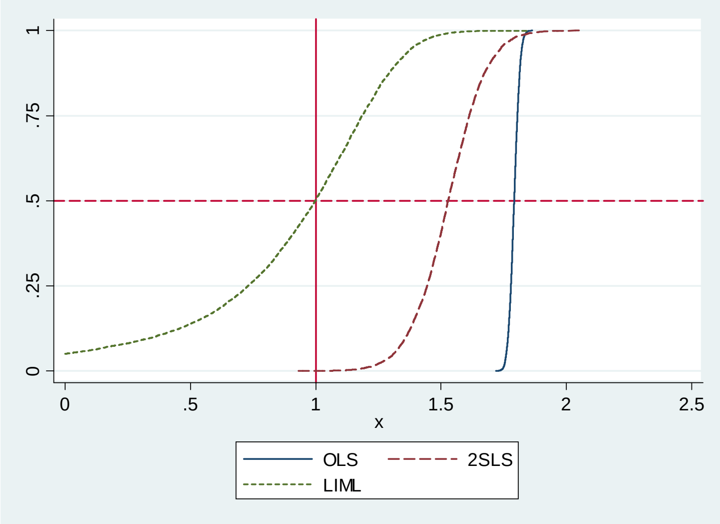
# Monte Carlo Results

## 2SLS, LIML: 2 instruments, IV: one instrument



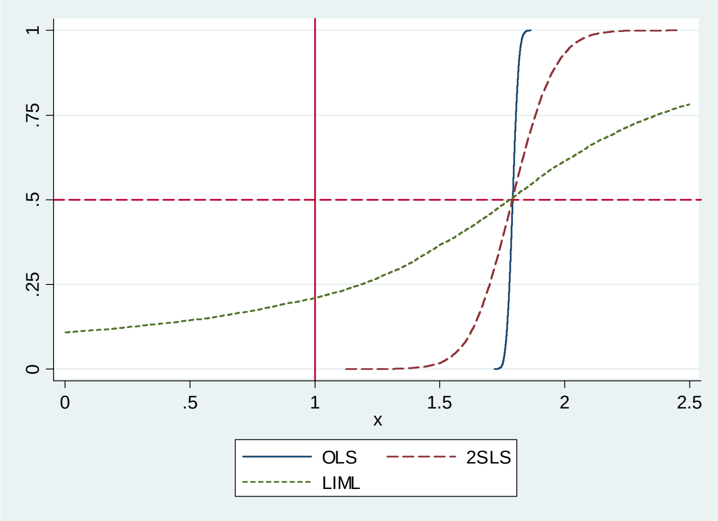
# Monte Carlo Results

## 20 instruments



# Monte Carlo Results

## 20 garbage instruments



# What should you do in practice?

- Report the first stage and think about whether it makes sense. Are the magnitude and sign as you would expect?
- Report the $F$-statistic on the excluded instruments. The bigger this is, the better. $F$s above 10 to 20 are considered relatively safe, lower $F$s put you in the danger zone.
- Pick your best single instrument and report just-identified estimates using this one only. Just-identified IV is approximately median-unbiased.
- Check over-identified 2SLS estimates with LIML. If the LIML estimates are very different, or standard errors are much bigger, worry.
- Look at the coefficients, $t$-statistics, and $F$-statistics for excluded instruments in the reduced-form regression of dependent variables on instruments. The reduced-form estimates are just OLS, so they are unbiased. If the relationship you expect is not in the reduced form, it's probably not there.

# Some alternative AK91 estimates

TABLE 4.6.2
Alternative IV estimates of the economic returns to schooling

| | (1) | (2) | (3) | (4) | (5) | (6) |
|---|---|---|---|---|---|---|
| 2SLS | .105 | .435 | .089 | .076 | .093 | .091 |
| | (.020) | (.450) | (.016) | (.029) | (.009) | (.011) |
| LIML | .106 | .539 | .093 | .081 | .106 | .110 |
| | (.020) | (.627) | (.018) | (.041) | (.012) | (.015) |
| *F*-statistic (excluded instruments) | 32.27 | .42 | 4.91 | 1.61 | 2.58 | 1.97 |
| *Controls* | | | | | | |
| Year of birth | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| State of birth | | | | | ✓ | ✓ |
| Age, age squared | | ✓ | | ✓ | | ✓ |
| *Excluded instruments* | | | | | | |
| Quarter-of-birth dummies | ✓ | ✓ | | | | |
| Quarter of birth*year of birth | | | ✓ | ✓ | ✓ | ✓ |
| Quarter of birth*state of birth | | | | | ✓ | ✓ |
| Number of excluded instruments | 3 | 2 | 30 | 28 | 180 | 178 |

# Violations of the exclusion restriction

- In general, we do not believe that any of the assumptions we are making in statistics are literally true. But we typically proceed if we think our assumptions are "pretty good."
- Is "pretty good" enough for the exclusion restriction? Suppose we have a candidate instrument $Z_i$ for a regressor $D_i$. Write

$$\begin{aligned} y_i &= \alpha + \rho D_i + \gamma Z_i + e_i \\ D_i &= \pi_0 + \pi_1 Z_i + \xi_i. \end{aligned}$$

The exclusion restriction amounts to the assumption $\gamma = 0$.

# Weak instruments and the exclusion restriction

What happens if $\gamma = \gamma_0 \neq 0$?

$$\begin{aligned}\rho_{IV} &= \frac{cov\left(\alpha + \rho D_i + \gamma_0 Z_i + e_i, Z_i\right)}{cov\left(D_i, Z_i\right)} \\ &= \frac{\rho cov\left(D_i, Z_i\right) + \gamma_0 var\left(Z_i\right)}{cov\left(D_i, Z_i\right)} \\ &= \rho + \gamma_0 \frac{var\left(Z_i\right)}{cov\left(D_i, Z_i\right)} = \rho + \frac{\gamma_0}{\pi_1}\end{aligned}$$

- The IV estimate of $\rho$ is biased, the bias is $\gamma_0/\pi_1$.
- The bias is larger in absolute value the closer $\pi_1$ is to zero. I.e. the bias is worse with weak instruments.
- With $\pi_1$ very close to zero, a very small violation of the exclusion restriction can lead to a large (asymptotic) bias.

# Can violation of the exclusion restriction explain AK91?

- School starting age (SSA) is a candidate violation of the exclusion restriction. Individuals born in quarter 4 start school younger.
- The best paper on school starting age is Black, Devereux, and Salvanes, "Too Young to Leave the Nest?" *REStat* 2011 for Norway. They find an effect on log earnings of -0.1 at age 25, -0.01 (not significant) at age 30, 0.0 at age 35, suggesting SSA works through lost labor market experience. (AK91 sample is in their 40s.)
- To play with the numbers:
  - $\pi_1 = 0.09$, this is the difference in schooling between quarter 4 and other quarters
  - Those born in quarter 4 start school about 6 months younger, or -0.5 of a year.
  - $\gamma_0$ is the effect of school starting age on earnings. Using the age 30 $\gamma_0$ of -0.01

$$\frac{\gamma_0}{\pi_1} = \frac{-0.01 * -0.5}{0.09} = 0.055$$

# Violations of random assignment

Suppose control $X_i$ is necessary for random assignment of instrument $Z_i$.

$$\begin{aligned} y_i &= \alpha + \rho D_i + \gamma X_i + e_i \\ D_i &= \pi_0 + \pi_1 Z_i + \pi_2 X_i + \xi_i . \end{aligned}$$

What happens if $\gamma \neq 0$ but we omit $X_i$?

$$\begin{aligned} \rho_{IV} &= \frac{cov\left(\alpha + \rho D_i + \gamma X_i + e_i, Z_i\right)}{cov\left(D_i, Z_i\right)} \\ &= \frac{\rho cov\left(D_i, Z_i\right) + \gamma cov\left(X_i, Z_i\right)}{cov\left(D_i, Z_i\right)} \\ &= \rho + \gamma \frac{cov\left(X_i, Z_i\right)}{cov\left(D_i, Z_i\right)} \end{aligned}$$

This looks like and is an omitted variables bias formula. The standard OLS coefficient $cov\left(X_i, D_i\right) / var\left(D_i\right)$ is being replaced by it's IV analogue: for the auxiliary regression run $X_i$ on $D_i$ and instrument by $Z_i$.

# Can violation of random assignment explain AK91?

- Magnitude: The bias is equal to

$$\gamma \frac{cov(X_i, Z_i)}{cov(D_i, Z_i)} = \gamma \frac{E[SES_i | Q4_i = 1] - E[SES_i | Q4_i = 0]}{E[S_i | Q4_i = 1] - E[S_i | Q4_i = 0]}.$$

  - $\gamma$ is the effect of $SES_i$ on child's earnings. Intergenerational correlations are about 0.4.
  - The top is the difference in $SES_i$; BJB report family income between first and other quarter births differs by about 2%, so this is 0.02.
  - The bottom is 1st stage with one instrument, 0.09.

$$\gamma \frac{E[SES_i | Q4_i = 1] - E[SES_i | Q4_i = 0]}{E[S_i | Q4_i = 1] - E[S_i | Q4_i = 0]} = 0.4 \frac{0.02}{0.09} = 0.088$$

- Pattern: Buckles and Hungerman (2008): Data on mother's education and month of birth.
  - Patterns don't line up: kid's schooling peaks in quarter 4 (Oct - Dec), mother's schooling peaks in May.