Public Provision of Private Goods and the Redistribution of Income

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Technical Appendix

The purpose of this Appendix is to provide formal verification of the assertions made in the text. We begin by showing that our assumptions on the utility function and demands imply that the marginal rate of substitution is decreasing in q and increasing in z.

Fact 1: Under our assumptions, $u_q(q,z)/u_z(q,z)$ is decreasing in q and increasing in z on $(\underline{q},\overline{q}) \times (0,\infty)$.

Proof: Let $(q^o,z^o) \in (\underline{q},\overline{q}) \times (0,\infty)$ be given. It is enough to show that

(A.1) $$u_z(q^o,z^o)u_{qq}(q^o,z^o) - u_{zq}(q^o,z^o)u_q(q^o,z^o) < 0,$$

and that

(A.2) $$u_z(q^o,z^o)u_{qz}(q^o,z^o) - u_{zz}(q^o,z^o)u_q(q^o,z^o) > 0.$$

Since preferences are strictly convex there must exist some price–income pair (p^o,y^o) such that $q(p^o,y^o) = q^o$ and $z(p^o,y^o) = z^o$. Because of our normality assumptions we know that $z_y(p^o,y^o) > 0$ and $q_y(p^o,y^o) > 0$. In the usual fashion these derivatives can be calculated to be:

(A.3) $$z_y(p^o,y^o) = u_z(q^o,z^o)[u_{qq}(q^o,z^o)u_z(q^o,z^o) - u_q(q^o,z^o)u_{qz}(q^o,z^o)]/\Delta,$$

and

(A.4) $$q_y(p^o,y^o) = u_z(q^o,z^o)[u_{zz}(q^o,z^o)u_q(q^o,z^o) - u_z(q^o,z^o)u_{qz}(q^o,z^o)]/\Delta,$$

where $\Delta \equiv u_{zz}u_q^2 + u_{qq}u_z^2 - 2u_{qz}u_qu_z < 0$ (by strong quasi–concavity). Now note that (A.1) follows from (A.3) and the fact that $z_y > 0$ and (A.2) follows from (A.4) and the fact that $q_y > 0$. □

Next we show that if the government confines its quality choice to the interval

$[q_g^{min}, q_g^{max}]$ only low income individuals will consume in the public sector.

Fact 2: Let $q_g \epsilon (q_g^{min}, q_g^{max})$, then

(i) $$u(q_g, y_L - \gamma_L p q_g) > v(p, y_L - \gamma_L p q_g),$$

and

(ii) $$u(q_g, y_H - \gamma_L p q_g) < v(p, y_H - \gamma_L p q_g).$$

Proof: (i) If $q_g \geq q(p, y_L - \gamma_L p q_g)$ the result is obvious, so we may assume that $q_g < q(p, y_L - \gamma_L p q_g)$. By definition,

$$u(q_g^{min}, y_L - \gamma_L p q_g^{min}) \geq v(p, y_L - \gamma_L p q_g^{min}).$$

Since $q_g > q_g^{min}$, it follows that

$$u(q_g^{min}, y_L - \gamma_L p q_g^{min}) > v(p, y_L - \gamma_L p q_g),$$

and thus it suffices to show that

$$u(q_g, y_L - \gamma_L p q_g) \geq u(q_g^{min}, y_L - \gamma_L p q_g^{min}).$$

To verify this inequality note first that

(A.5) $$v(\gamma_L p, y_L) \geq u(q_g^{min}, y_L - \gamma_L p q_g^{min}),$$

and that

$$q(\gamma_L p, y_L) \geq q(p, y_L - \gamma_L p q_g) > q_g > q_g^{min}.$$

Now let $\lambda \epsilon (0,1)$ be such that $q_g = \lambda q(\gamma_L p, y_L) + (1-\lambda) q_g^{min}$, then using (A.5) and the fact that $u(q,z)$ is quasi–concave, we have

$$\begin{aligned} u(q_g, y_L - \gamma_L p q_g) &= u(\lambda q(\gamma_L p, y_L) + (1-\lambda) q_g^{min}, y_L - \gamma_L p(\lambda q(\gamma_L p, y_L) + (1-\lambda) q_g^{min})) \\ &\geq u(q_g^{min}, y_L - \gamma_L p q_g^{min}). \end{aligned}$$

(ii) Since $q_g < q_g^{max}$ it suffices to show that

$$u(q_g, y_H - \gamma_L p q_g) \leq v(p, y_H - \gamma_L p q_g^{max}).$$

Suppose not. Then

(A.6) $$u(q_g, y_H - \gamma_L p q_g) > v(p, y_H - \gamma_L p q_g^{max}).$$

Now let $\hat{q} = q(p,y_H-\gamma_L pq_g^{max})$. Since, by definition,

(A.7) $$u(q_g^{max},y_H-\gamma_L pq_g^{max}) \leq v(p,y_H-\gamma_L pq_g^{max}),$$

we know that $\hat{q} > q_g^{max} > q_g$. Choose $\lambda \in [0,1]$ such that $\lambda\hat{q} + (1-\lambda)q_g = q_g^{max}$. Then by (A.6) and quasi–concavity

$$\begin{aligned} v(p,y_H-\gamma_L pq_g^{max}) &\leq u(\lambda\hat{q}+(1-\lambda)q_g,\lambda(y_H-\gamma_L pq_g-p\hat{q})+(1-\lambda)(y_H-\gamma_L pq_g)) \\ &= u(q_g^{max},y_H-\gamma_L p(\lambda\hat{q}/\gamma_L+(1-\lambda)q_g+\lambda q_g)) \\ &< u(q_g^{max},y_H-\gamma_L pq_g^{max}), \end{aligned}$$

which contradicts (A.7). □

We now demonstrate that the interval $[q_g^{min},q_g^{max}]$ will be non–empty if the rich are not attracted to the public sector by free provision of the minimal quality level.

Fact 3: Suppose that

(A.8) $$v(p,y_H-\gamma_L p\underline{q}) > u(\underline{q},y_H-\gamma_L p\underline{q}),$$

then $q_g^{min} < q_g^{max}$.

Proof: Condition (A.8) implies that $q_g^{max} > \underline{q}$. We may therefore assume that $q_g^{min} > \underline{q}$, which implies that

(A.9) $$u(q_g^{min},y_L-\gamma_L pq_g^{min}) = v(p,y_L-\gamma_L pq_g^{min}).$$

To prove the fact it therefore suffices to show that

(A.10) $$u(q_g^{min},y_H-\gamma_L pq_g^{min}) < v(p,y_H-\gamma_L pq_g^{min}),$$

since then, by continuity, $q_g^{max} > q_g^{min}$.

Let $\hat{q} = q(p,y_L-\gamma_L pq_g^{min})$ then we know from (A.9) that $\hat{q} > q_g^{min}$. Moreover

$$v(p,y_H-\gamma_L pq_g^{min}) \geq u(\hat{q},y_H-p\hat{q}-\gamma_L pq_g^{min})$$

and hence it is enough to show that

$$u(\hat{q},y_H-p\hat{q}-\gamma_L pq_g^{min}) > u(q_g^{min},y_H-\gamma_L pq_g^{min}).$$

To this end, define the function $\mu(q)$ from the equation

$$u(q,y_L-\mu) = u(q_g^{min},y_L-\gamma_L p q_g^{min}).$$

Note that $\mu(q_g^{min})=\gamma_L p q_g^{min}$, $\mu(\hat{q})=p\hat{q}+\gamma_L p q_g^{min}$, and

$$\mu'(q) = u_q(q,y_L-\mu)/u_z(q,y_L-\mu).$$

By the Fundamental Theorem of Calculus we may write

$$u(\hat{q},y_H-p\hat{q}-\gamma_L p q_g^{min})-u(q_g^{min},y_H-\gamma_L p q_g^{min}) = \int_{q_g^{min}}^{\hat{q}} f'(q)dq,$$

where $f(q) \equiv u(q,y_H-\mu(q))$. But

$$f'(q) = u_z(q,y_H-\mu(q))[u_q(q,y_H-\mu(q))/u_z(q,y_H-\mu(q)) - u_q(q,y_L-\mu(q))/u_z(q,y_L-\mu(q))]$$

which is positive since $u_q(q,z)/u_z(q,z)$ is increasing in z by Fact 1. It follows that $\int_{q_g^{min}}^{\hat{q}} f'(q)dq$ is positive which proves the desired inequality. □

Our next task is to verify the assertion in footnote 5 that q_g^* exists and is unique if poor individuals would choose to consume the indivisible good if given a cash transfer of $p\underline{q}$ and required to pay taxes of $\gamma_L p\underline{q}$.

Fact 4: Suppose that $q(p,y_L+p\underline{q}-\gamma_L p\underline{q}) \neq 0$, then q_g^* exists and is unique.

Proof: Define the function

$$h(q_g) = q_g - q(p,y_L+pq_g-\gamma_L pq_g). \tag{A.11}$$

It suffices to show that the equation $h(q_g) = 0$, has a unique solution. Since $q(p,y_L+p\underline{q}-\gamma_L p\underline{q}) \neq 0$, h is continuous on $[\underline{q},\overline{q}]$, $h(\underline{q}) \leq 0$, and $h(\overline{q}) \geq 0$. It follows from the Intermediate Value Theorem that there exists a solution to the equation $h(q_g) = 0$. To show that it is unique we show that $h(\cdot)$ is increasing on $[\underline{q},\overline{q}]$. Differentiating (A.11) we obtain

$$h'(q_g) = 1 - q_y(p,y_L+pq_g-\gamma_L pq_g)p(1-\gamma_L).$$

Since $z(p,y) + pq(p,y) = y$ and z is a normal good, we know that

$$q_y(p,y_L+pq_g-\gamma_L pq_g)p = 1 - z_y(p,y_L+pq_g-\gamma_L pq_g) \le 1.$$

Thus $h'(q_g) > 0$ as required. □

It is worth noting that if $q(p,y_L+p\underline{q}-\gamma_L p\underline{q}) = 0$, then q_g^* does not exist. So the condition is both necessary and sufficient for existence.

We now verify the second and third properties claimed for the equivalent variation function.

Fact 5: $m(q_g) = pq_g$ if and only if $q_g = q_g^*$.

Proof: From equation (4) in the text we see that

$$m(q_g) = pq_g \text{ iff } e(p,u(q_g,y_L-\gamma_L pq_g)) = y_L+pq_g-\gamma_L pq_g$$

But

$$e(p,u(q_g,y_L-\gamma_L pq_g)) = y_L+pq_g-\gamma_L pq_g \text{ iff } u(q_g,y_L-\gamma_L pq_g) = v(p,y_L+pq_g-\gamma_L pq_g)$$

and by strong quasi–concavity

$$u(q_g,y_L-\gamma_L pq_g) = v(p,y_L+pq_g-\gamma_L pq_g) \text{ iff } q_g = q(p,y_L+pq_g-\gamma_L pq_g). \quad \square$$

Fact 6: For all $q_g \in [q_g^{min},q_g^*]$

$$m'(q_g) > \gamma_L p.$$

Proof: Differentiating equation (4) in the text we obtain

$$m'(q_g) = e_u(\cdot)[u_q(q_g,y_L-\gamma_L pq_g)-\gamma_L pu_z(q_g,y_L-\gamma_L pq_g)] + \gamma_L p.$$

Thus it suffices to show that

$$u_q(q_g,y_L-\gamma_L pq_g)/u_z(q_g,y_L-\gamma_L pq_g) > \gamma_L p$$

for q_g in the relevant range. To see this, first let

$$\phi(q_g) = u_q(q_g,y_L-\gamma_L pq_g)/u_z(q_g,y_L-\gamma_L pq_g).$$

We know that $\phi(q_g^*) = p$, and hence it suffices to show that $\phi'(q_g) < 0$. Define the function

$$\sigma(q,z) = u_q(q,z)/u_z(q,z).$$

Note that $\sigma_q < 0$ and $\sigma_z > 0$ by Fact 1 and that

(A.12) $$\phi(q_g) = \sigma(q_g, y_L - \gamma_L p q_g).$$

Differentiating (A.12) we obtain

$$\phi'(q_g) = \sigma_q - \gamma_L p \sigma_z < 0,$$

as required. □

Our final task is to demonstrate that the feasibility frontier with universal public provision can never cross the 45^o line.

Fact 7: For all $q_g \in [q_g^{min}, q_g^{max}]$

$$v(p, y_H - \gamma_L p q_g) > v(p, y_L + m(q_g) - \gamma_L p q_g).$$

Proof: Let $q_g \in [q_g^{min}, q_g^{max}]$. Then

$$\begin{aligned} v(p, y_L + m(q_g) - \gamma_L p q_g) &= u(q_g, y_L - \gamma_L p q_g) \\ &< u(q_g, y_H - \gamma_L p q_g) \\ &\leq v(p, y_H - \gamma_L p q_g), \end{aligned}$$

where the last inequality follows from Fact 2. □

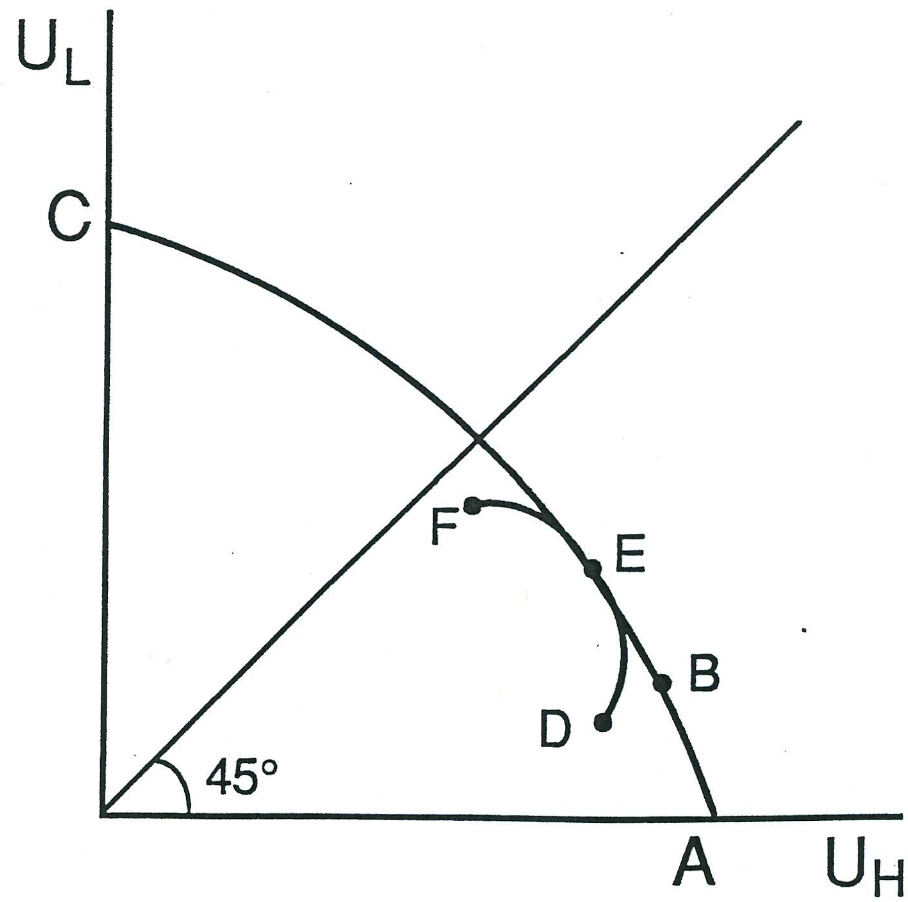

FIGURE 1

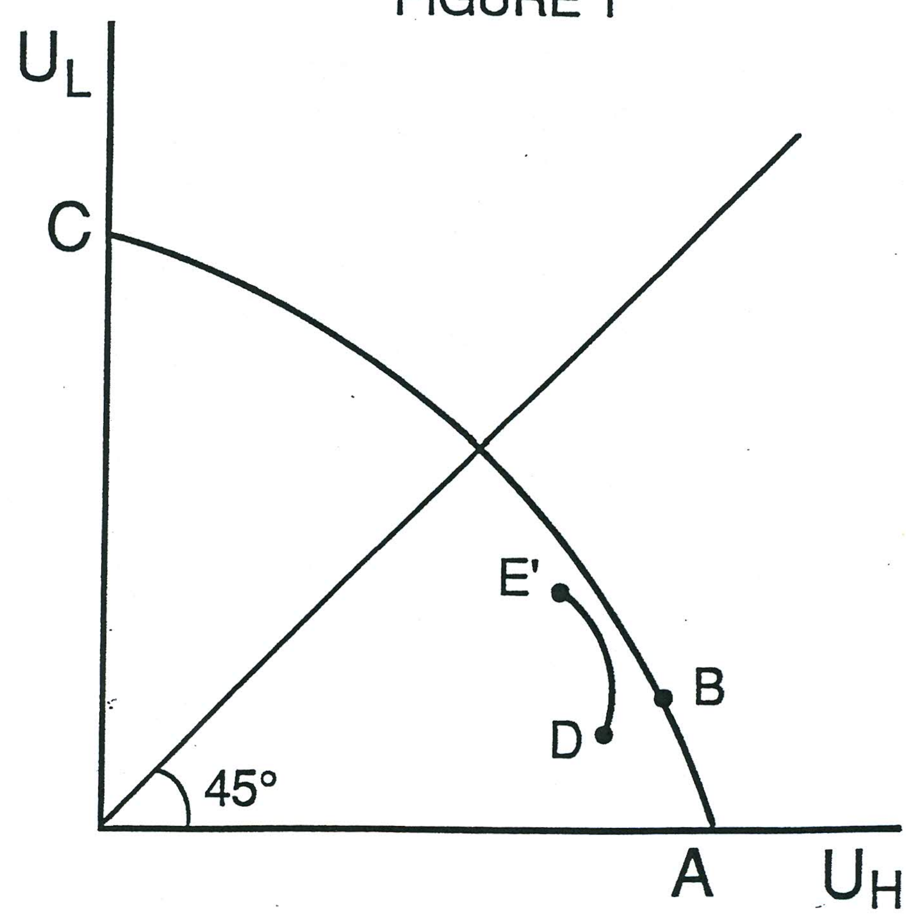

FIGURE 2