- Topic 13. Six illustrative cases of hypothesis testing in linear regression

  Preliminary discussion: Auxiliary regression procedure for the LM/score test

  Main discussion: Examining each of the six cases one-by-one

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Estimator	Notation	
1.	$\hat{\beta}_{OLS}$	
2.	$\hat{\beta}_{LAD}$	$= MLE$ with LDE errors
3.	$\hat{\beta}_{Lstar}$	needed for optimality results
4.	$\hat{\beta}_{GMM}$	$= OLS$
$5a.$	$\hat{\beta}_{IGLS}$	
$5b.$	$\hat{\beta}_{FGLS}$	
6.	$\hat{\beta}_{MLE}$	

Summary of the six leading cases

- Case 1: NLRM.$A4GM(iid)$ or NLRM.$A4\Omega$ plus linear restrictions on β^{true}
  - Note 1: "U" model is NLRM
  - Note 2: Given all restrictions are linear, we can re-write "R" model to also be NLRM
- Case 2: NLRM.$A4GM(iid)$ or NLRM.$A4\Omega$ plus non-linear restrictions on β^{true}
  - Note: "U" model is NLRM
- Case 3: NLRM.$A4\Omega.CondHeterosk$ vs. NLRM.$A4GM(iid)$
  - Note: "R" model is NLRM
- Case 4: NLRM.$A4\Omega.AR2$ vs. NLRM.$A4GM(iid)$
  - Note: "R" model is NLRM
- Case 5: $A1, A2Linear, A3.ExogEndog, A4GM(iid)or\Omega$, and $A5Gaussian$ vs. same assumptions except we swap in $A3.StronglyExog$
  - Note: "R" model is NLRM
- Case 6: $A1, A2Linear, A3.StronglyExog, A4GMiid$, and $A5LinearExponentialFamilyIndexedBy\lambda$ vs. $A5Gaussian$ (corresponding to $\lambda = 2$)
  - Note: "R" model is NLRM

Trinity III: LM (aux.reg.)

Consider any inferential case wherein the "R" model is NLRM.$A4GM(iid)$ (in which case, obviously OLS is the best estimation strategy). In such a case, a major shortcut exists for implementing the LM test. This is typically referred to as the "auxiliary regression" approach:

- **Step 1:** Estimate the "R" Model via OLS using X^R, the regressor matrix that is appropriate for the "R" model and thus obtain $\hat{\theta}^R_{OLS}$. Save the vector of OLS residuals, $\hat{\varepsilon}^R_{OLS}$.
- **Step 2:** Consider the auxiliary regression (to be also estimated by OLS) as follows:
  - Dependent variable (LHS): $\hat{\epsilon}^R_{OLS}$ or a suitable function of the residuals (to be explained when we discuss the six leading cases in detail).
  - Explanatory variables (RHS): X^R and additional variables appearing in the "U" model over and above the "R" model.
  - Save the R^2 from this auxiliary regression as R^2_{aux}.
- **Step 3:** Calculate $S \cdot R^2_{aux}$ (where S is sample size) and use the result that under H_0:

$$SR^2_{aux} \xrightarrow{d} \chi^2_r \text{ as } S \to \infty.$$

Summary of the six leading cases

- Case 1: NLRM.$A4GM(iid)$ or NLRM.$A4\Omega$ plus linear restrictions on β^{true}
  - Note 1: "U" model is NLRM
  - Note 2: Given all restrictions are linear, we can re-write "R" model to also be NLRM, and aux.reg. approach for LM test should work

- Case 2: NLRM.$A4GM(iid)$ or NLRM.$A4\Omega$ plus non-linear restrictions on β^{true}
  - Note: "U" model is NLRM, but "R" model is *not* NRLM, so aux.reg. approach for LM test will *not* work

- Case 3: NLRM.$A4\Omega.CondHeterosk$ vs. NLRM.$A4GM(iid)$
  - Note: "R" model is NLRM, and aux.reg. approach for LM test should work

- Case 4: NLRM.$A4\Omega.AR2$ vs. NLRM.$A4GM(iid)$
  - Note: "R" model is NLRM, and aux.reg. approach for LM test should work

- Case 5: $A1, A2Linear, A3.ExogEndog, A4GM(iid)or\Omega$, and $A5Gaussian$ vs. same assumptions except we swap in $A3.StronglyExog$
  - Note: "R" model is NLRM, and aux.reg. approach for LM test should work

- Case 6: $A1, A2Linear, A3.StronglyExog, A4GMiid$, and $A5LinearExponentialFamilyIndexedBy\lambda$ vs. $A5Gaussian$ (corresponding to $\lambda = 2$)
  - Note: "R" model is NLRM, and aux.reg. approach for LM test should work

Case 1: NLRM with linear restrictions on β^{true}

	"R" model (Maintained plus H_0)	"U" model (Maintained)
Maintained:	$A1, A2linear, \geq A3Rmi, A4GM(iid), A5Gaussian$	$\leftarrow$ same
Null/Alt.:	H_0: $R\beta^{true} = q$	$H_1 : R\beta^{true} = ?$
Best estim.:	RLS: $\min(RSS)$ s.t. $R\beta^{true} = q$ (e.g., X_{partB} are 0)	OLS for $(\beta^{true}, \sigma^2_\varepsilon)'$
Notes:	We could impose restrictions and do OLS (not RLS)	
	Wald: work with $R\hat{\beta}_{OLS} - q$	
	LR: consider RSS_{RLS} vs. RSS_{OLS} or likelihoods	
	LM: shortcut "aux.reg." is possible	

- Note: Since OLS is best for the "R" model, the aux.reg. shortcut is possible.
  - Step 1: Reg. y on restricted set of X variables, say X_{partA}, and save residuals e^R_{OLS}.
  - Step 2: Reg. e^R_{OLS} on original X^R. That is, both X_{partA} and X_{partB}, the latter of which was omitted from the "R" model.
  - Step 3: Calculate $S \cdot R^2_{aux}$, which under H_0, converges (in dist.) as $S \to \infty$ to a χ^2_r random variable (e.g., here $r = k_B$). Thus, we reject H_0 if the auxiliary regression exhibits a good fit.

Case 2: NLRM with non-linear restrictions on β^{true}

	"R" model (Maintained plus H_0)	"U" model (Maintained)
Maintained:	$A1, A2linear, \geq A3Rmi, A4GM(iid), A5Gaussian$	$\leftarrow$ same
Null/Alt.:	H_0: $g(\beta^{true})= q$	$H_1 : g(\beta^{true})=?$
Best estim.:	RLS: $\min(RSS)$ s.t. $g(\beta^{true}) = 0$	OLS for $(\beta^{true}, \sigma_\epsilon^2)'$
Notes:	In general, not possible to impose restrictions	
	Wald: work with $g(\hat{\beta}_{OLS})$	
	LR: consider RSS_{RLS} vs. RSS_{OLS} or likelihoods	
	LM: shortcut "aux.reg." not possible	
	All final results are asymptotic (non-linearity)	

- Note: Since OLS is not best for the "R" model, the aux.reg. shortcut is not possible.

Case 3: Conditional heteroskedasticity tests in NLRM

	"R" model (Maintained plus H_0)	"U" model (Maintained)
Maintained:	$A1, A2linear, \geq A3Rmi, ...$ [see next]	$\leftarrow$ swap in $A4\Omega.CondHeterosk$
	$A4GM(iid)$, and $A5Gaussian$	$\mathbb{E}(\varepsilon_s^2 \vert X) = \delta_0 + \delta_1 x_{s3}^2 + \delta_2 x_{s5}^2$
Null/Alt.:	H_0: $g(\beta^{true}) = q$	$H_1 : \delta_1 = ?, \delta_2 = ?$
Best estim.:	OLS	FGLS or MLE for... [see next]
		$(\beta^{true}, \delta_0, \delta_1, \delta_2)'$
Notes:	Wald: work with $(\hat{\delta}_{1,FGLS/MLE}, \hat{\delta}_{2,FGLS/MLE})$	
	LR: consider $\log \mathcal{L}_{FGLS}$ vs. $\log \mathcal{L}_{OLS}$	
	LM: shortcut "aux.reg." is possible	
	All final results are asymptotic (FGLS)	

- Note: Since OLS is best for the "R" model, the aux.reg. shortcut is possible.
  - Step 1: Reg. y on X and save residuals, $e^R_{OLS,s}$ for $s = 1, ..., S$.
  - Step 2: Use $(e^R_{OLS,s})^2$ as the LHS for estimating the given skedastic equation via OLS.
  - Step 3: Calculate $S \cdot R^2_{aux}$, which under H_0, converges (in dist.) as $S \to \infty$ to a χ^2_r random variable (e.g., here $r = 2$). Thus, we reject H_0 if the auxiliary regression exhibits a good fit.

Case 4: Autocorrelation tests in NLRM

	"R" model (Maintained plus H_0)	"U" model (Maintained)
Maintained:	$A1, A2linear, \geq A3Rmi, ...$	$\leftarrow$ swap in $A4\Omega.AR2$
	$A4GM(iid)$, and $A5Gaussian$	$\varepsilon_s = \gamma_1\varepsilon_{s-1} + \gamma_2\varepsilon_{s-2} + \nu_s, ...$
		$\nu\|X \sim N(0, \sigma^2_\nu I_S)$
Null/Alt.:	$H_0 : \gamma_1 = \gamma_2 = 0$	$H_1 : \gamma_1 {=} ?, \gamma_2 {=} ?$
Best estim.:	OLS	FGLS or MLE for...
		$(\beta^{true}, \sigma^2_\nu, \gamma_1, \gamma_2)'$
Notes:	Wald: work with $(\hat{\gamma}_{1,FGLS/MLE}, \hat{\gamma}_{2,FGLS/MLE})$	
	LR: consider $\log \mathcal{L}_{FGLS}$ vs. $\log \mathcal{L}_{OLS}$	
	LM: shortcut "aux.reg." is possible	
	All final results are asymptotically valid only	

- Note: Since OLS is best for the "R" model, the aux.reg. shortcut is possible.
  - Step 1: Reg. y on X and save residuals, $e^R_{OLS,s}$ for $s = 1, ..., S$.
  - Step 2: Use $(e^R_{OLS,s})^2$ as the LHS for estimating the given $AR(2)$ equation via OLS.
  - Step 3: Calculate $S \cdot R^2_{aux}$, which under H_0, converges (in dist.) as $S \to \infty$ to a χ^2_r random variable (e.g., here $r = 2$). Thus, we reject H_0 if the auxiliary regression exhibits a good fit.

Cases 5 and 6

For these two final cases, please see slides below (which have been re-used from past years).

12.5 *Case 5: Regressor Exogeneity*

	R Model (Maintained plus H_0)	U Model (Maintained)
Maintained:	A1+A2linear+A3RZsru +A4GM+ S very large	same
	A2: $y = X^B\beta_x^B + X^G\beta_x^G + \epsilon_{true} = X^B\beta_x^B + Z^I\beta_z^I + \epsilon_{true}$,	$X^B = Z^I\pi^I + Z^E\pi^E + u$
	$A3RZsru: Ez_s\epsilon_s^{true} = 0$ and $Ez_su_s = 0$	for Z^I and Z^E
H_0 and H_1:	$H_0: A3RX^Bsru: Ex_s^B\epsilon_s^{true} = 0 \quad r = k_B$	$Ex_s^B\epsilon_s^{true} \neq 0$ possibly
Best estim:	OLS of y on all X (i.e., X^B and Z^I)	IVE/2SLS using RF of X^B on all Z (i.e
		to get β_x, β_z^I, π^I, and π^E
Notes:	Wald: $\widehat{corr}(x_s^B, \epsilon_s^{true})$ vs. $0_{k^B\times 1}$	
	LR: $LLF_{ive/2sls}$ vs. LLF_{ols}	
	LM: Shortcut "Auxiliary Reg" possible:	
	significance of $\hat{u}_{ols}$ in OLS of y on X and Z^I	
	All final results are Asymptotic due to IV/2SLS	

NB: Since best R model estimation is OLS, LM-Auxiliary Regression shortcut is possible

Step 1: Run OLS of y on original X (i.e., X^B and X^G=Z^I) and save the residuals e_ols

Step 2: Regress e_ols on X (i.e., X^B and X^G=Z^I) *and* Z^E. Equivalently, on X (i.e., X^B and X^G=Z^I) *and* the OLS RF residuals u_ols

Step 3: Calculate SxRsquared from Aux regression. For very large sample size S, under H_0 SxRsquared will be chi-squared(r) degrees of freedom — here r=k_B. Thus, we reject H_0 if the Aux regression has a good fit.

12.6 *Case 6: Nonnormality of the Errors*

	R Model (Maintained plus H_0)	U Model (Maintained)
Maintained:	A1+A2linear+A3Rmi or > +A4GM+A5Stable distrn	with parameter λ^{true}
H_0 and H_1:	$H_0 : A5Gaussian\ which\ is\ equivalent\ to\ \lambda^{true} = 2$	λ^{true}=?
Best estim	OLS	MLE to get β_{true} and λ
Notes:	Wald: work with $\hat{\lambda}_{mle} - 2$	
	LR: $LLF_{U:MLE}$ vs. $LLF_{R:OLS}$	
	LM: shortcut "Auxiliary Reg" possible	
	using higher order polynomials of errors	

NB: Since best R model estimation is OLS, LM-Auxiliary Regression shortcut is possible:

Step 1: Run OLS of y on X and save the residuals e_ols

Step 2: Regress e_ols on X and higher order polynomials of e_ols.

Step 3: Calculate SxRsquared from Aux regression. For very large sample size S, under H_0 SxRsquared will be chi-squared(r) degrees of freedom — here r=1. Thus, we reject H_0 if the Aux regression has a good fit.