London School of Economics
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Answers for Problem Set #6

Question 1

Explain what we mean by the "Normal Linear Regression Model (NLRM)".

NLRM-A4GM: $\hat{\beta}_{ols}|X \sim N(\beta^{true}, \sigma^2(X'X)^{-1})$

NLRM-A4Omega: $\hat{\beta}_{ols}|X \sim N(\beta^{true}, c^2(X'X)^{-1}X'\Omega X(X'X)^{-1})$

Outline the conditions that characterize this model.

NLRM-A4GM: $A1 + A2 + A3Rmi$ or stronger $A3 + A4GM(iid) + A5Gaussian$

NLRM-A4Omega: $A1 + A2 + A3Rmi$ or stronger $A3 + A4\Omega + A5Gaussian$

Repeat with the term "Asymptotic Linear Regression Model (ANLRM)".

Part 1: If "A" is taken to mean "Approximate NLRM for very large, *finite* sample size S"

ANLRM-A4GM: $\hat{\beta}_{ols}|X \approx N(\beta^{true}, \sigma^2(X'X)^{-1})$

ANLRM-A4Omega: $\hat{\beta}_{ols}|X \approx N(\beta^{true}, c^2(X'X)^{-1}X'\Omega X(X'X)^{-1})$

Outline the conditions that characterize this model.

ANLRM-A4GM: $A1 + A2 +$ $A3Rsru$ or stronger (i.e., *any* of the 7 versions of $A3) + A4GM(iid) + S$ very large.

ANLRM-A4Omega: $A1 + A2 + A3Rsru$ or stronger (i.e., *any* of the 7 versions of $A3) + A4\Omega + S$ very large.

Part 2: If "A" is taken to mean "Asymptotic NLRM for sample size S growing unboundedly large (i.e., limits as $S \to \infty$)"

ANLRM-A4GM:

$$\sqrt{S}(\hat{\beta}_{ols} - \beta^{true}) \underset{S \to \infty}{\overset{d}{\to}} N\left(0, \sigma^2 \left(p\lim\left(\frac{X'X}{S}\right)\right)^{-1}\right)$$

ANLRM-A4Omega:

$$\sqrt{S}(\hat{\beta}_{ols} - \beta^{true}) \underset{S \to \infty}{\overset{d}{\to}} N\left(0, c^2\left(\left(p\lim\left(\frac{X'X}{S}\right)\right)^{-1}\right) p\lim\left(\frac{X'\Omega X}{S}\right)\left(\left(p\lim\left(\frac{X'X}{S}\right)\right)^{-1}\right)\right)$$

Outline the conditions that characterize this model.

ANLRM-A4GM: $A1 + A2 +$ $A3Rsru$ or stronger (i.e., *any* of the 7 versions of $A3) + A4GM(iid) + S \to \infty$.

ANLRM-A4Omega: $A1 + A2 + A3Rsru$ or stronger (i.e., *any* of the 7 versions of $A3) + A4\Omega + S \to \infty$.

1. Based on the NLRM, discuss which quantities will be distributed as $\chi^2(\cdot)$ and which as $F(\cdot, \cdot)$.

   These distributional results are discussed in the Slides and Extended Notes of weeks 5-8 and will not be repeated here.

2. Will the same two distributions be relevant for the case of the ANLRM?

   As $S \to \infty$, the random variables that are in the denominators of the t-statistic and of the F-statistic will converges to something fixed — this is because the denominators contain the estimated s^2_{ols} which converges asymptotically to the true σ^2 — you can also see this from the fact that the t-statistic will be distributed $t(S-k)$ and the F-statistic will be $F(r, S-k)$. As $S \to \infty$,

   $t(S-k) \to t(\infty) = N(0,1)$

   $F(r, S-k) \to F(r, \infty) = \chi^2(r)/r$

   In practice, econometricians recommend still using the $t(.)$ and $F(.,.)$ distributions even for very large S, because the resulting rejection regions are more conservative compared to their $N(0,1)$ and $\chi^2(r)$ approximations.

Question 2

This question is given as an illustration of testing non-linear hypotheses in the lecture Slides and Extended Notes.
Using the $g(\cdot)$ notation, this implies:

$$g(\beta) = \begin{pmatrix} \beta_1\beta_2 - 1 \\ \beta_3 - 4\beta_4 + 2 \end{pmatrix} \text{ and } g_\beta(\beta) = \begin{pmatrix} \beta_2 & \beta_1 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{pmatrix}.$$

Hence, in this case the test statistic (based on Abraham Wald's principles) corresponds to:

$$W = g(\hat{\beta})' \left(g_\beta(\hat{\beta}) \hat{V}(\hat{\beta}) g_\beta(\hat{\beta})' \right)^{-1} g(\hat{\beta})$$

$$= \left(\hat{\beta}_1\hat{\beta}_2 - 1, \hat{\beta}_3 - 4\hat{\beta}_4 + 2 \right) \left(\begin{pmatrix} \hat{\beta}_2 & \hat{\beta}_1 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{pmatrix} \hat{V}(\hat{\beta}) \begin{pmatrix} \hat{\beta}_2 & 0 \\ \hat{\beta}_1 & 0 \\ 0 & 1 \\ 0 & -4 \end{pmatrix} \right)^{-1} \begin{pmatrix} \hat{\beta}_1\hat{\beta}_2 - 1 \\ \hat{\beta}_3 - 4\hat{\beta}_4 + 2 \end{pmatrix}.$$

The test statistic will be asymptotically distributed as $\chi^2(2)$ under H_0 under assumptions (A1), (A2), (A3Rcu), and (A4). Note in particular that the weakest (A3) is allowed (contemporaneously uncorrelated ϵ and regressors) and (A5Normality) is *not* required.

Since this statistic is asymptotically distributed in the limit when $S \to \infty$, it will not be valid for any finite sample size. One could use these asymptotic distributional results as an approximation, provided the finite sample size S is very large.

Question 3

ANSWERS:

Consider the classic linear regression model, where A1+A2+A3Rmi hold.

Reminder: $A1 : rank(X) = k$, $A2 : y = X\beta^{true} + \epsilon^{true}$ with $E(\epsilon^{true}) = 0$, and $A3Rmi : E(\epsilon^{true}|X) = \epsilon^{true}$.

It is also believed that: $A4 : E(\epsilon^{true}\epsilon^{true\prime}|X) = c^2\Omega(\lambda)$.

1. Define the IGLS estimator for β^{true} for the case that the λ parameter vector is known, and the FGLS estimator for β^{true} for the case that λ^{true} is unknown but a preliminary estimator $\hat{\lambda}$ for it is available.

   $\hat{\beta}_{igls} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$

   $\hat{\beta}_{fgls} = (X'\Omega(\hat{\lambda})^{-1}X)^{-1}X'\Omega(\hat{\lambda})^{-1}y$

   where $\hat{\lambda} = \hat{\lambda}(y, X)$. The typical requirement is that the preliminary (or joint) estimator of λ^{true}, $\hat{\lambda}$, is a $\sqrt{S}$-consistent estimator in the sense that:

   $$\sqrt{S}(\hat{\lambda} - \lambda^{true}) \underset{S\to\infty}{\overset{d}{\to}} N(0, VCov(\hat{\lambda}))$$

2. Derive the sampling error vectors for IGLS and FGLS.

   By using $A2 : y = X\beta^{true} + \epsilon^{true}$ into the IGLS definition, we obtain:

   $\hat{\beta}_{igls} - \beta^{true} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\epsilon^{true}$

   Using A2 into the FGLS definition, however, yields a non-linear function of y, because $\hat{\lambda} = \hat{\lambda}(y, X)$:

   $\hat{\beta}_{fgls} - \beta^{true} = (X'\Omega(\hat{\lambda}(y, X)^{-1}X)^{-1}X'\Omega(\hat{\lambda}(y, X))^{-1}y - \beta^{true} = NLF(y, X, \beta^{true})$

   So this representation is useless in the sense that (a) the sampling error vector is a Non-Linear Function of y, and (b) it depends on the unknown β^{true}.

3. Use the sampling error vectors from (b) to explain the major differences in the properties of the two estimators.

   IGLS:

   Unbiased? For known Ω, conditional on X it is clear that IGLS is unbiased (which means also unbiased) since the conditional expectation passes through to ϵ^{true} — which is 0 from A3Rmi or stronger and the second part of A2.

   Consistent? The key requirement will be that $\frac{1}{S}X'\Omega^{-1}\epsilon^{true} \underset{S\to\infty}{\overset{p}{\to}} 0$ which will hold under $A1 - A4\Omega$ under suitable technical conditions for Law of Large Numbers (LLN); and also that $\frac{1}{S}X'\Omega^{-1}X \underset{S\to\infty}{\overset{p}{\to}} M_{X\Omega invX}$ finite, non-singular limiting $k \times k$ matrix.

Asymptotically normal? In addition to $\frac{1}{S}X'\Omega^{-1}X \underset{S\to\infty}{\overset{p}{\rightarrow}} M_{X\Omega invX}$, we now require that

$$\frac{1}{\sqrt{S}}X'\Omega^{-1}\epsilon^{true} \underset{S\to\infty}{\overset{d}{\rightarrow}} N\left(0, p\lim\left(\frac{1}{S}X'\Omega^{-1}X\right)\right)$$

which will hold under $A1 - A4\Omega$ under suitable technical conditions for Central Limit Theorem (CLT)

FGLS:

Unbiased? The fact that $\hat{\beta}_{fgls} - \beta^{true} = NLF(y, X, \beta^{true})$ means that FGLS will be *biased* in general *even* *if* A3Rmi or stronger holds.

Consistent and Asymptotically Normal?

There are two approaches to establish this:

Method 1: Prove that $\hat{\lambda} \underset{S\to\infty}{\overset{p}{\rightarrow}} \lambda^{true}$, which makes FGLS converge to the IGLS one, whose Consistency and Asymptotic Normality were established above. OR (which is technically more "clean")

Method 2: Prove that

(a)

$$p\lim\frac{1}{S}\left(X'\left(\Omega^{-1} - \hat{\Omega}^{-1}\right)X\right)^{-1} \underset{S\to\infty}{\overset{p}{\rightarrow}} 0$$

(b)

$$p\lim\frac{1}{S}\left(X'\left(\Omega^{-1} - \hat{\Omega}^{-1}\right)\epsilon^{true}\right) \underset{S\to\infty}{\overset{p}{\rightarrow}} 0$$

(c)

$$p\lim\frac{1}{\sqrt{S}}\left(X'\left(\Omega^{-1} - \hat{\Omega}^{-1}\right)\epsilon^{true}\right) \underset{S\to\infty}{\overset{p}{\rightarrow}} 0$$