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Answers for Problem Set #8

Question 1

Consider the "Truncated" regression model with exogenous regressors, defined by:

$$y_s = \begin{cases} x_s'\beta + \epsilon_s & if \quad x_s'\beta + \epsilon_s > 0 \\ -999 & otherwise \end{cases}$$

The sample is indexed by $s = 1, \cdots, S$ and the error term ϵ_s is distributed i.i.d.$N(0, \sigma^2)$ conditional on the regressors X. It can be shown that the following result holds about the distribution of $y|X$:

Result 1:

$$E(y_s|X, x_s'\beta + \epsilon_s > 0) = x_s'\beta + \sigma \frac{\phi\left(\frac{x_s'\beta}{\sigma}\right)}{\Phi\left(\frac{x_s'\beta}{\sigma}\right)}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution respectively.

1. A colleague proposes to ignore Result1 and simply regress by OLS the observed y_s on the observed x_s' regressors. How well would you expect this method to perform in practice?

   ANSWER:

   Consider the simple linear regression:

   $$y_s = x_s'\beta + u_s$$

   where the composite error $u_s \equiv y_s - x_s'\gamma$ has conditional expectation:

   $$Eu_s|X \equiv Ey_s|X - x_s'\gamma = x_s'\beta + \sigma \frac{\phi\left(\frac{x_s'\beta}{\sigma}\right)}{\Phi\left(\frac{x_s'\beta}{\sigma}\right)} - x_s'\beta = \sigma \frac{\phi\left(\frac{x_s'\beta}{\sigma}\right)}{\Phi\left(\frac{x_s'\beta}{\sigma}\right)} \neq 0$$

   So the regressors x_s' will *NOT* be exogenous w.r.t. the composite error u_s, which means that the run regression will violate A3. Hence, the naive OLS will be inconsistent for the true β.

2. A colleague proposes to use Result 1 and estimate β and sigma by the intuitively obvious extension to OLS given the nonlinear regression function on the RHS:

   ANSWER:

Using Result 1 means that we exploit the function of the expectation of the observed y and x_s' , i.e., we can define the non-linear regression equation:

$$y_s = x_s'\beta + \sigma \frac{\phi\left(\frac{x_s'\beta}{\sigma}\right)}{\Phi\left(\frac{x_s'\beta}{\sigma}\right)} + v_s$$

where v_s will have 0 conditional mean given X. Therefore, the NLLS estimators for β and σ are defined as:

$$\begin{pmatrix} \hat{\beta} \\ \hat{\sigma} \end{pmatrix}_{NLLS} = \arg\min_{\beta,\sigma} \sum_{s \text{ for } y_s>0} \left(y_s - x_s'\beta - \sigma \frac{\phi\left(\frac{x_s'\beta}{\sigma}\right)}{\Phi\left(\frac{x_s'\beta}{\sigma}\right)} \right)^2$$

3. Which proposal, (a) or (b), do you think is more likely to produce better estimates? Explain intuitively your answer.

   ANSWER: the NLLS in (b) has a *major* advantage over the OLS of part (a) in that the NLLS in (b) will be CUAN, whereas the plain OLS in (a) will be *inconsistent*.

   NB: the naive approach (a) is an example of what I call the PHP approach for LDV models: Pretend, Hope, and Pray!

Question 2

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$$s^2 = \frac{T}{T-K}\left[\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2 - \frac{1}{T}\sum_{t=1}^{T}x_t'\epsilon_t\left(\frac{\sum_{t=1}^{T}x_tx_t'}{T}\right)^{-1}\frac{1}{T}\sum_{t=1}^{T}x_t\epsilon_t\right]$$

1. as $T \to \infty$, $\lim \frac{T}{T-K} = 1$
2. since $\epsilon_t \tilde{} iid\left(0, \sigma_\varepsilon^2\right)$ by Khinchine's LLN $\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2 \to_p \to \sigma_\varepsilon^2 < \infty$ because $\epsilon_t^2 \tilde{} iid$ with mean $E\epsilon_t^2 = \sigma_\varepsilon^2 < \infty$
3. Since $x_t \tilde{} iid\left(\mu_x, \Sigma_x\right)$, $\frac{\sum_{t=1}^{T}x_tx_t'}{T} \to_p \to \Sigma_x$ because by Khinchine x_tx_t' will be iid with mean $Ex_tx_t' = \Sigma_x < \infty$
4. Consider the weakest case: $Ex_t\epsilon_t = 0$ (i.e contemporaneously uncorrelated), but $x_t\epsilon_t$ may be correlated with $x_s\epsilon_s, t \neq s$

   By Chebyshev Markov's LLN $\frac{\sum_{t=1}^{T}x_t\epsilon_t}{T} \to_{MSE} \to 0$ (which implies $\frac{\sum_{t=1}^{T}x_t\epsilon_t}{T} \to_p \to 0$) as long as $Ex_t\epsilon_t = 0$ which holds by assumption and $\frac{1}{T^2}V\left(\sum_{t=1}^{T}x_t\epsilon_t\right) \to_{T\to\infty} \to 0$ But $\frac{1}{T^2}V\left(\sum_{t=1}^{T}x_t\epsilon_t\right) = \frac{1}{T^2}\sum_{t=1}^{T}Ex_tx_t'\epsilon_t^2$ since $Ex_t\epsilon_t = 0$ and $E\left(\epsilon_s\epsilon_t|X\right) = 0$. By the assumptions $E\left(\epsilon\epsilon'|X\right) = \sigma_\varepsilon^2 I_T$ there is no conditional heteroskedasticity $\left(E(\epsilon_t^2|x_s) = \sigma_\varepsilon^2\right)$ and therefore $Ex_tx_t'\epsilon_t^2 = \sigma_\varepsilon^2 Ex_tx_t' = \sigma_\varepsilon^2\Sigma_x$.Then $\frac{1}{T^2}V\left(\sum_{t=1}^{T}x_t\epsilon_t\right) = \frac{1}{T^2}T\sigma_\varepsilon^2\Sigma_x \to_{T\to\infty} \to 0$

   Putting these results through Slutsky's Theorem allows us to conlclude that under these conditions $s^2 \to_p \to \sigma_\varepsilon^2$.

   Alternatively, using matrix notation, $\frac{1}{T^2}V\left(\sum_{t=1}^{T}x_t\epsilon_t\right) = \frac{1}{T^2}V(X'\epsilon) = \frac{1}{T^2}E(X'\epsilon\epsilon')$. Thus if $E\left(\epsilon\epsilon'|X\right) = \sigma_\varepsilon^2 I_T$,then $\frac{1}{T^2}V\left(\sum_{t=1}^{T}x_t\epsilon_t\right) = \frac{1}{T}\sigma_\varepsilon^2\Sigma_x$.