DUALITY, CONSUMPTION DECISIONS UNDER UNCERTAINTY, AND LIQUIDITY CONSTRAINTS: A Note * **

by

Vassilis A. Hajivassiliou
Cowles Foundation
Yale University
New Haven, CT 06520-2125
Telephone: 203-432-3699
E-mail: vassilis@econ.yale.edu

Yannis M. Ioannides
Department of Economics
VPI&SU
Blacksburg, VA 24061-0316
Telephone: 703-231-7981
E-mail: yannis@vtvm1.cc.vt.edu

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Abstract

A dual approach to the problem of maximizing lifetime utility subject to liquidity constraints in discrete time leads to a dual dynamic programming formulation which links commodity and asset demand theory under uncertainty with Frisch demand theory. We provide a full characterization of the dependence of the underlying Lagrange multiplier upon the fundamentals of the problem. We establish the existence of a threshold level of wealth, which characterizes constrained behavior. We explore the consequences of the dual approach for empirical work for both decisions of individuals and firms.

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Address editorial correspondence to:
Yannis M. Ioannides
Department of Economics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061-0316, USA.

1 INTRODUCTION

Maximization of the expectation of a time-separable lifetime utility index that ranks lifetime consumption bundles, subject to well-defined investment opportunities, lends itself neatly to a dynamic programming formulation [Samuelson (1969); Merton (1969; 1971); Bewley (1977)]. Yet, it was only relatively recently that the problem with a general utility function in continuous time was solved explicitly [Karatzas, Lehoczky, Sethi and Shreve (1986)].[1] Here we assume individuals receive endowments in the form of labor income, but are prohibited from borrowing against that income: they may not hold negative wealth.[2] A dual approach enables us to interpret the optimal solution of the liquidity-constrained problem as an unconstrained problem under an implicit, *individual–specific,* set of Arrow-Debreu state prices.[3]

Under uncertainty, duality theory confers the additional advantage that the partial differential Hamilton-Jacoby equation obtained by dynamic programming for the dual problem in continuous time is linear and thus easier to solve than the well-known one for the primal problem.

In the present paper we suitably adapt new research tools developed primarily by finance theorists who have explored decisions under uncertainty in the presence of alternative sets of institutional restrictions, such as incompleteness of asset markets and prohibition of short sales. We show that a dual framework provides a firm link with the theory of marginal utility of wealth-constant demand functions, or *Frisch* demands [Browning, Deaton and Irish (1985)]. This extension has been shown by applied theorists to be essential for empirical applications of demand theory in dynamic settings. This connection has not been made previously and is explored further here. We characterize the optimal policy by means of a threshold value of assets, below which expenditure exhausts all beginning-of-period assets.

[1] See Zeldes (1989b) and Deaton (1991) for numerical approaches.

[2] Dybvig and Huang (1988) prove that a nonnegativity-of-wealth constraint precludes arbitrage opportunities when quite general trading strategies are considered in continuous time models. See also Cox and Huang (1989), Dybvig and Huang (1988), and Karatzas, Lehoczky and Shreve (1987).

On a bit of history, Cox and Huang was available as a working paper as of 1986. Karatzas *et al.* were unaware of this work and worked independently. They differ from Cox and Huang by their avoidance of L_2 theory. Our ability to rule out getting something for nothing by imposing a constraint that may be institutionally motivated is particularly welcome. Somewhat more general is the approach of He and Pearson (1991a,b) who assume incomplete markets and include short-sale constraints. Karatzas, Lehoczky, Shreve and Xu (1991) go further than He and Pearson by using duality theory and local martingale methods developed by Xu (1990) and Xu and Shreve (1990). See Karatzas, *et al.* (1991) for further details on the relationship of their work with He and Pearson's work.

[3] See Artle and Varayia (1978), Pissarides (1978), and Jackman and Sutton (1982) who were the first to show in deterministic settings that the optimal consumption policy takes the form of a sequence of subproblems defined over subhorizons where consuming all income as it is received alternates with optimizing over several subperiods.

Our model allows us to characterize fully the informational requirements of Frisch demand theory in the presence of liquidity constraints.

Extensions to standard problems of optimization by firms readily follow and some implications for econometric modelling are also pursued further. The dual variable of interest in the case of firms is interpreted as the marginal value of capital, known as *marginal* q, which under certain identifying assumptions [Hayashi (1982)] may be observable. Such a linkage with the $q-$theory of investment enhances the attractiveness of duality theory. In either case, duality leads to stochastic difference equations which is linear in the unknown (dual) variable.

Our work is closely related to Deaton (1991) and, in addition, combines ideas from He and Pages (1993) and He and Pearson (1991a; 1991b). Both Deaton (1991), who uses discrete time, and He and Pages (1993), who use continuous time, articulate the threshold value of wealth but work with only a scalar decision variable. He and Pearson (1991a) employ a discrete-time state-space representation of uncertainty to analyze consumption and asset demands but do not touch upon the threshold value of assets. Our emphasis on the usefulness of duality in solving a particular problem in finite time also distinguishes our paper from Bewley (1977) [4] and Chamberlain and Wilson (1984), both of whom emphasize limiting results when the number of periods tends to infinity. The linkage with the $q-$theory of investment is straightforward but new.

2 THE MODEL

We consider decisions of individuals who live finite lifetimes of length T in an economy that runs for a countable number of discrete periods $t \in \{0, 1, \ldots, T\}$. Uncertainty in this economy is characterized by means of a probability space $(\boldsymbol{\Omega}, \boldsymbol{N}, \boldsymbol{P})$, where an element $\omega \in \boldsymbol{\Omega}$ stands for a particular realization of the relevant random variables in this economy from 1 to T. Information in this economy is represented by a sequence of partitions of $\boldsymbol{\Omega}$, $\{N_t \mid t = 0, 1, \ldots, T\}$. The interpretation of this information structure is that at time t the agent knows which cell of N_t contains the true state. Information increases through time; N_{t+1} is at least as fine as N_t. It is standard to take N_0 as the trivial partition, $N_0 = \boldsymbol{\Omega}$, and N_T the discrete partition, i.e., $N_T = \{\omega \mid \omega \in \boldsymbol{\Omega})$. The σ-field of events generated by N_t is denoted by $\boldsymbol{\mathcal{N}}_t$, and $\boldsymbol{N} = \{\boldsymbol{\mathcal{N}}_t;\ t \in \{0, 1, \ldots, T\}\}$ is the filtration generated by the sequence of partitions $\{N_t; t \in \{0, 1, \ldots, T\}\}$. $\mathbf{P}$ is the associated probability measure; for

[4] Bewley (1977) allows for many commodities and a much more general stochastic structure than ours, but assumes only a single riskless asset, money.

any event e in $\boldsymbol{N}$, its probability is $P(e)$. This information structure is rather standard and may be easily and intuitively represented by an event tree. The representation of uncertainty according to filtration $\boldsymbol{N}$ is finite dimensional, that is $\boldsymbol{\Omega}$ is finite. We use E_t to denote the expectations operator associated with $\boldsymbol{\mathcal{N}}_t$.

A typical individual faces a wage rate $W_{h,t}$ and a vector of prices of all other goods, $W_{g,t}$, and is characterized by an exogenous maximum amount of leisure in period t, $\bar{L}_t$. The vector $W_t = (W_{h,t}, W_{g,t})$ is a stochastic process that is adapted to the filtration $\boldsymbol{\mathcal{N}}$. Let $\bar{L}_t$, the endowment of leisure in period t, be a stochastic process that is also adapted to $\boldsymbol{N}$. We use H_t to denote the value of the endowment of leisure: $H_t \equiv W_{h,t}\bar{L}_t$. The randomness of prices is assumed to introduce no additional source of uncertainty.

Let $u_t(\ell_t, G_t \mid \mathcal{N}_t)$ denote a utility function as a function of leisure ℓ_t and a vector of other goods, G_t and be conditional on all new information available to the individual as of time t, $\mathcal{N}_t$. Such a general dependence allows for taste shocks. Utility per period, $u_t(\ell_t, G_t \mid \cdot)$ is assumed to be concave, twice-differentiable and increasing with respect to all of its arguments (ℓ_t, G_t). To $u_t(\cdot \mid \cdot)$ there corresponds an indirect utility function

$$v_t(b; W \mid \mathcal{N}_t) = \max_{\{(\ell,G):\, b=W_h\ell+W_g'G\}} : u_t(\ell, G \mid \mathcal{N}_t), \tag{1}$$

with the standard properties.

We introduce additional notation to describe the asset structure in the economy. We assume that there exist $K+1$ securities, $k \in \{0, 1, 2, \ldots, K\}$, that households may trade costlessly each period for current consumption and labor. We assume that security 0 is a riskless one-period bond. To simplify matters, we assume that K is finite [*c.f.* Altug and Miller (1990); He and Pearson (1991a)] and that all securities are one-period securities. Let $q_{k,t}$ denote the quantity of security k held by the individual from period $t-1$ to period t, $s_{k,t}$ its period t price, and assume for simplicity (and without loss of generality) that securities pay no dividends.

Let $\mathcal{N}_t$-measurable variables be defined as follows: F_t is the market value of *financial* and A_t the market value of *total* wealth at the beginning of period t (before expenditure decisions for that period are made), $F_t = \sum_{k=0}^{K} s_{k,t}q_{k,t}; \quad t = 1, \ldots, T.$; F_t^* is the end-of-period financial wealth (after expenditure decisions in period t have been incurred), $F_t^* = \sum_{k=0}^{K} q_{k,t+1}s_{k,t}; t = 1, \ldots, T.$; b_t is total expenditure on consumption and leisure, *full expenditure,* in period t. We then have:

$$b_t = W_{g,t}'G_t + W_{h,t}\ell_t. \tag{2}$$

The household's budget constraint in period t is:

$$A_t \equiv F_t + H_t = b_t + F_t^*; \quad t = 0, 1, \dots T. \tag{3}$$

The securities trading (portfolio) strategy is a predictable $(K+1)$-dimensional process $\boldsymbol{q}$, with $\boldsymbol{q} = \{(q_{0,t}, \dots, q_{K,t}), t = 1, \dots, T\}$, where predictable means that $(q_{0,t}, \dots, q_{K,t})$ is measurable with respect to $\mathcal{N}_{t-1}$. The $(K+1)$-dimensional process $\boldsymbol{s} = \{(s_{0,t}, \dots, s_{K,t}), \ \ t = 0, 1, \dots, T\}$, is $\mathcal{N}_t$-adapted.

The problem faced by the typical individual may now be restated as follows.

Problem 1. *Find a net asset decumulation path* $\boldsymbol{b} = \{b_t, 0 \leq t \leq T\}$, *and a portfolio strategy* $\boldsymbol{q} = \{(q_{0,t}, q_{1,t}, \dots, q_{K,t}) \ \ t = 0, 1, \dots, T-1\}$ *to maximize:*

$$v_t(b_t; W_t \mid \mathcal{N}_t) + E_t \left\{ \sum_{j=t+1}^{T} \frac{1}{(1+\rho)^{j-t}} v_j(b_j; W_j \mid \mathcal{N}_t) \right\}, \tag{4}$$

subject to constraints (3), to a given value for financial wealth at the beginning of the process, $F_0 \equiv \sum_{k=0}^{k=K} s_{k,0} q_{k,0}$, *and to a requirement that the individual be solvent at the end of period* T.

Once $\boldsymbol{b}$ has been found, the consumption of leisure and all other elements of the consumption bundle (ℓ_t, G_t) follows Roy's identity. In dynamic optimization problems existence and uniqueness are ensured provided that some regularity conditions are satisfied.[5] Here we are interested in exploring the economic intuition of the problem by means of assumptions which are much stronger than needed for existence and uniqueness [See Bhattacharya and Majumdar, *op. cit.*].

2.1 Complete vs. Incomplete Markets

It is well known that with complete markets, Problem (1) is vastly simplified. All trades, denoted by an $\mathcal{N}_t$-measurable vector (ℓ_t, G_t) and an $\mathcal{N}_{t-1}$-measurable vector $(q_{0,t}, \dots, q_{K,t})$ for each $t \in \{1, \dots, T-1\}$, occur at time period 0 and satisfy a *single* lifetime budget constraint. This constraint is obtained by requiring that the present value of lifetime expenditure minus the present value of lifetime receipts, both evaluated at the contingent claims prices, not

[5]The weakest such conditions are those of Bhattacharya and Majumdar (1989). They require the following: (A1) the state space be a nonempty Borel subset of a complete separable metric space; (A2) the choice set be a compact metric space for all possible values of the state variables; (A3) $v(\cdot)$ be uppersemicontinuous, if $v(\cdot)$ is bounded, and continuous otherwise, forall feasible values of the choice variables and of the decision variables; and $U(x) \equiv \sum_{k=0}^{\infty}(1+\rho)^{-t} v_k(x) < \infty$, forall x in the state space, where $v_0(x) \equiv \sup_{d \in C(x)} |v(x,d)|$, $v_{k+1}(x) \equiv \sup_{d \in C(x)} \int v_k(z) \pi(dz|x,d)$, $k = 0, 1, \dots,$ d denotes the decision and $C(x)$ denotes the choice set, and $\pi(dz \mid x, d)$ the law of motion; (A4) The law of motion $\pi(dz|x,d)$ is a weakly continuous function of (x,d), if $v(x,d)$ is bounded. Otherwise, the mapping $(x,d) \mapsto \int \phi(z) \pi(dz|x,d)$ must be continuous for all Borel measurable ψ satisfying $|\psi(x)| \leq U(x) + 1$, forall x in the state space.

exceed initial wealth F_0. Under complete markets, such a valuation is equivalent to taking expectations.

We use $\Lambda_t(\cdot)$, a measure defined on $\mathcal{N}_t$ for each $t \in \{0, 1, \ldots, T\}$, to denote prices of contingent claims defined in terms of the numeraire good in period t [Altug and Miller, *op. cit.*]. For example, for any event, represented by a set $e_t \in N_t$, the nonnegative real number $\Lambda_t(e_t)$ denotes the period 0 price of a unit of the numeraire good to be delivered on date t, contingent on event e_t occurring. The usual assumption is that Λ_t is absolutely continuous with respect to $\boldsymbol{P}$. This assumption implies a "density", denoted by $\frac{1}{(1+\rho)^t}\lambda_t$, of contingent claims prices, technically the Radon-Nikodym derivative of Λ_t with respect to $\boldsymbol{P}$. Hence, $\Lambda_t(e_t) = \frac{1}{(1+\rho)^t}\int_{e_t}\lambda_t(\omega)\boldsymbol{P}\,(d\omega)$. We may now write the lifetime budget constraint corresponding to (3) under complete markets as follows [*ibid*, p. 547]:

$$E_0\left\{\sum_{t=0}^{T}\frac{1}{(1+\rho)^t}\lambda_t\ [b_t - H_t]\right\} \leq A_0. \tag{5}$$

The necessary conditions for the maximization of (4), $t = 1, \ldots, T$, subject to (5) are:

$$\frac{\partial}{\partial b_t}v_t(b_t; W_t \mid \mathcal{N}_t) = \mu\lambda_t; \tag{6}$$

for all measurable sets of $\mathcal{N}_t$ and $\forall t \in \{1, \ldots, T\}$. μ is a Lagrange multiplier associated with the individual's lifetime budget constraint (5), reflecting individual characteristics, and λ_t is the (density of) contingent claims price, reflecting market conditions.

In the absence of complete markets, individuals may not be able to transfer purchasing power across all states of nature. The endowment of leisure may not be sold at the beginning of the process via the contingent-claims markets. Now trades take place as uncertainty evolves sequentially, and the sequence of budget constraints must be satisfied for all measurable subsets of $\mathcal{N}_t$ and $t \in \{0, 1, \ldots, T\}$. Analytically, the most tangible consequence is that the Lagrange multiplier in the corresponding first-order conditions, the counterpart of (6), does not assume that particular form. Instead, it satisfies the following conditions:

$$\gamma_t = \frac{1}{1+\rho}E_t\left\{\gamma_{t+1}\frac{s_{k,t+1}}{s_{k,t}}\right\};\ \ k = 0, 1, \ldots, K;\ t = 0, 1, \ldots, T. \tag{7}$$

where $\frac{1}{(1+\rho)^t}\gamma_t$ is the Radon-Nikodym derivative of the period t Lagrange multiplier associated with the individual's period t constraint.

Market incompleteness is reflected neatly on the difference between λ_t and γ_t. We elaborate further on this below. If markets are complete, the Lagrange multiplier γ_t assumes the special form $\gamma_t = \mu\lambda_t$. In that case, γ_t is multiplicatively separable with respect to individual characteristics, encapsulated in μ, and market information, expressed by the (market-dependent) density λ_t, defined in (7). [6] Otherwise, the Lagrange multiplier γ_t is a time-dependent random variable, with a domain in the dual space corresponding to the space where the consumption bundle, securities and prices are defined, that depends on individual characteristics and the economy's asset structure.

2.2 Liquidity Constraints

Liquidity constraints express the (rather reasonable) restriction that individuals may not borrow against their future labor income and consequently the individual's end-of-period assets may not become negative in any period. [7] Even if markets were complete in the sense of full spanning (all states of nature are accessible) imposition of liquidity constraints imposes a certain type of incompleteness. It is this type of incompleteness which is examined by He and Pagès (1993). With the notable exception of Altug and Miller (1990), who formally test and find evidence in favor of complete markets, and of Runkle (1991), who finds evidence in favor of the permanent income hypothesis, most other empirical research to date, including notably Zeldes (1989a), have provided evidence against unconstrained versions of the life cycle model. Similar conclusions may be drawn from the literature on decisions of firms.[8]

If individuals are not allowed to hold negative financial wealth at the end of period t, then the expenditure plan $\{b_t : \quad t = 0, 1, \ldots, T\}$ and $\{A_t : \quad t = 1, 2, \ldots, T\}$ must satisfy a sequence of liquidity (i.e., borrowing) constraints:

$$A_t - b_t \geq 0, \qquad t = 0, 1, \ldots, T. \tag{8}$$

That is, with probability one the individual can never run into debt. We will not restrict trading in assets in any other way. It is possible, in particular, that the individual can short one asset and long another asset. Therefore, condition (8) translates into a requirement that the present value in terms of contingent-claims prices of expenditure for all possible paths leading up all realizations of the relevant random variables in every period must not exceed the present value of the endowment of leisure plus initial financial assets. Note that since assets left at the end of the lifetime horizon are not valued by the individual, (8) implies $A_T = b_T$. In contrast, in the complete markets case, where constraints (8) are absent, that

[6]The multiplicative separability of the Lagrange multiplier in the complete markets case lies at the heart of the innovative estimation procedure in Altug and Miller (1990).

[7]Before one prohibits entirely trading of the income endowment process, one may consider trading the expected value of the endowment process, that is by allowing borrowing against that expected value. An analogue is pursued by Clarida (1987) who examines the possibility that one may borrow against the lower support of the distribution of labor income.

[8]Please see Bond and Meghir (1994) and sources cited therein .

is the only constraint. Accounting for liquidity constraints is simplified by the notation we introduce below.

3 LIQUIDITY CONSTRAINTS – PRIMAL APPROACH

Our results are summarized in a number of propositions, which are presented in this section. Our assumption that the representation of uncertainty is finite allows us to deal with optimization problem defined in a finite dimensional vector space. This assumption may be relaxed at the cost of additional mathematical complication, which is not strictly necessary.

We now introduce some additional notation. Let n_t denote the number of cells in N_t, $n_t = \mid N_t \mid$ and let $L = \sum_{t=0}^{T} \mid N_t \mid$. Any $\boldsymbol{N}$-adapted process can be characterized by its values at these L cells. It is most convenient to allocate the first coordinate to date 0's cell, the next n_1 coordinates to date 1's cells, the next n_2 coordinates to date 2's cells, and so on. [see He and Pearson (1991a)].

Any bundle of full expenditures $\boldsymbol{b}$ may be fully characterized by a vector $\boldsymbol{B} \in \mathbb{R}^L$. Similarly, the value of the endowment of leisure may also be represented by a vector $\boldsymbol{H} \in \mathbb{R}^L$. Under the assumption that all asset markets are open each period then $M = (1+K)\sum_{t=0}^{T-1} \mid N_t \mid$ is the total number of dimensions needed to represent all possible trades in assets. Let $\boldsymbol{Z}$ denote the $L \times M$ matrix representing all investment opportunities. Rows represent some all possible outcomes (states of nature) in all periods. Columns represent payoffs of all assets in all states of nature. A dynamic trading strategy $\boldsymbol{Q} = \{q_t,\ t = 1, \ldots, T\}$, where q_t is the $(K+1)$-dimensional vector $(q_{0,t}, \ldots, q_{K,t})$, may be accordingly be represented by a vector in $\mathbb{R}^M$. Recall that $q_{k,t}$ is $\mathcal{N}_{t-1}$-measurable, that is, $q_{k,t}$ is set in period $t-1$ and denotes assets purchased in $t-1$ and liquidated in t. We will refer to it as $q_k(\nu_{t-1}; t)$.

Let ν_t denote the generic element in N_t. The elements of N_{t+1} that may be reached from ν_t are denoted by $N_{t+1} \cap \nu_t$. The opportunity to invest in security k in event ν_t generates an element equal to $-s_k(\nu_t)$ in the ν_t-th row of $\boldsymbol{Z}$, which corresponds to buying security k in ν_t, and element(s) $\{s_k(j);\ j \in N_{t+1} \cap \nu_t\}$ in the respective rows of $\boldsymbol{Z}$, which correspond to the cost of closing out the position next period. The first $1+K$ columns of $\boldsymbol{Z}$ are allocated to investment strategies in period 0, the next $1+K$ to the investment strategies at the first node of N_1, etc. For the riskless security $k = 0$, we define the one-period riskless rate of return, which may be time-dependent, as follows: $1 + \sigma_{t+1} \equiv \frac{s_0(\nu_{t+1})}{s_0(\nu_t)}$.

The expenditure bundle produced by a dynamic trading strategy $\boldsymbol{Q}$ is given by:

$$\boldsymbol{B} = \boldsymbol{Z}\boldsymbol{Q} + \boldsymbol{H}. \tag{9}$$

Various restrictions on trades may be represented conveniently in this setting. If certain securities are not available for trading at certain states (or, more generally, markets are incomplete) then the matrix Z would be missing certain columns. Prohibition of short-sales constraints may be represented by imposing nonnegativity on the relevant components of $\boldsymbol{Q}$.

The market value of total wealth in a state ν_t, as defined in (3), may be written in the new notation as

$$A(\nu_t) = \sum_{k=0}^{K} q_{k,t}(\nu_{t-1}) s_k(\nu_t) + h(\nu_t), \;\; \forall \nu_t \in N_t, \; t = 0, 1, \ldots, T. \tag{10}$$

Liquidity constraints according to (8) may now be expressed as :

$$\sum_{k=0}^{K} q_{k,t}(\nu_{t-1}) s_k(\nu_t) + h(\nu_t) - b(\nu_t) \geq 0, \;\; \forall \nu_t \in N_t, \; t = 0, 1, \ldots, T, \tag{11}$$

where suitable adjustments are made if some of the $K+1$ securities are not available for trading in some state and period. Let $(1+\rho)^{-t}\pi(\nu_t)$ denote the Lagrange multiplier corresponding to the ν_t-th row of (9). Let $(1+\rho)^{-t}\zeta(\nu_t)$ denote the Lagrange multipliers corresponding to (11), respectively, which are construed as $\mathcal{N}_t$-adapted processes. The Lagrange multipliers $(1+\rho)^{-t}\pi(\nu_t)$ and $(1+\rho)^{-t}\zeta(\nu_t)$ are positively valued components of L-dimensional vectors that belong to the dual of $\mathbb{R}^L$, that is they belong to $\mathbb{R}^L_+$.

Proposition 1 offers necessary and sufficient conditions for an equivalent statement of Problem 1, subject to liquidity constraints (11).

Proposition 1.

(a) Problem 1, subject to liquidity constraints, may be restated as

$$\min_{\{\pi(\cdot),\zeta(\cdot)\}} : \left\{ \max_{\{b(\cdot),q(\cdot)\}} : \sum_{t=0}^{T} \sum_{\nu_t \in N_t} (1+\rho)^{-t} p(\nu_t) \left\{ \left[\, v_t(b(\nu_t); W(\nu_t) \mid \mathcal{N}_t) - (\pi(\nu_t) + \zeta(\nu_t))\, b(\nu_t) \right] , \right. \right.$$
$$+\zeta(\nu_t) \left[\sum_{k=0}^{K} q_{k,}(\nu_{t-1}) s_k(\nu_t) + h(\nu_t) \right]$$
$$\left. \left. +\pi(\nu_t) \left[\sum_{k=0}^{K} \left(\left[q_{k,t}(\nu_{t-1}) - q_{k,t+1}(\nu_t) \right] s_k(\nu_t) + h(\nu_t) \right) \right] \right\} \right\}. \tag{12}$$

(b) Necessary and sufficient conditions for Problem (12) are: (9) and

$$\frac{\partial}{\partial b(\nu_t)} v_t(b(\nu_t); W(\nu_t) \mid \mathcal{N}_t) = \pi(\nu_t) + \zeta(\nu_t); \tag{13}$$

$$\pi(\nu_t) = (1+\rho)^{-1} \sum_{\nu_{t+1} \in N_{t+1} \cap \nu_t} p(\nu_{t+1} \mid \nu_t) \, [\pi(\nu_{t+1}) + \zeta(\nu_{t+1})] \, \frac{s_k(\nu_{t+1})}{s_k(\nu_t)}, \forall \nu_t \in N_t, t = 0, 1, \ldots, T-1, \quad (14)$$

$\forall k \in \{0, 1, \ldots, K\}$. *If (11) holds as a strict inequality,* $\zeta(\nu_t) = 0$ *and the individual is* *unconstrained* *in period t; otherwise* $\zeta(\nu_t) > 0$ *and the individual is* *constrained* *in period t.*

Proof. The restatement of Problem 1 according to (a) above readily follows if we adjoin constraints (9) and (11) and multiply each one by $p(\nu_t)$. Thus the adjoined state-by-state constraints may be brought under the summation operators. Then (b) follows by rewriting and applying the Kuhn-Tucker theorem [Luenberger (1969)].

□ □

This result corresponds neatly to Proposition 4.1 in He and Pearson (1991a), p.7, except that we do not assume absence of arbitrage. The $\pi(\cdot)$'s in (14) are elements of $\mathbb{R}^L_+$, the same space as the (unique) state price vector when market are complete, but are now individual-specific. In contrast, He and Pearson (1991a) require that asset prices would be martingales under the corresponding equivalent measure. It is therefore clear that the presence of liquidity constraints complicates the problem. If (11) never hold as equalities, then all $\zeta(\nu_t) = 0$ and we are back to the complete markets case. The problem is separable in terms of the $\pi(\nu_t)'$s, which are determined by conditions (14). These conditions yield a unique state price vector, if markets are complete, which is normalized in terms of $\pi(\nu_0)$ and depends upon the asset structure only. $\pi(\nu_0)$ plays the role of μ in (6).

3.1 Asset Demands

As a byproduct of this analysis we may use duality to characterize asset demands. Specifically, let $V(A(\nu_t); W(\nu_t) \mid \mathcal{N}_t)$ the value function of dynamic programming, defined as the optimal value obtained from the maximization of argument of the max operator in (12) above, when truncated at the respective points in time. Maximization is subject to constraints (10) and (11), as adjoined according to (13) and to (14), except that we invoke (10) to express remaining utility as a function of $A(\nu_t)$ and of the other state variables. That is :

$$V[A(\nu_t); W(\nu_t); s_0(\nu_t), \ldots, s_K(\nu_t) \mid \mathcal{N}_t] =$$

$$\min_{\{\pi(\cdot),\zeta(\cdot)\}} : \left\{ \max_{\{b(\cdot),q(\cdot)\}} : \{[\, v_t(b(\nu_t); W(\nu_t) \mid \mathcal{N}_t) - (\pi(\nu_t) + \zeta(\nu_t)) \, b(\nu_t)] \right.$$

$$+ \left[\zeta(\nu_t)A(\nu_t) + \pi(\nu_t)\right] A(\nu_t) - \pi(\nu_t) \sum_{k=0}^{K} q_{k,t+1}(\nu_t) s_k(\nu_t)$$

$$+\frac{1}{1+\rho} \sum_{\nu_{t+1} \in N_t \cap \nu_t} p(\nu_{t+1} \mid \nu_t) V \left[\sum_{k=0}^{K} q_k(\nu_t; t+1) s_k(\nu_{t+1}); W(\nu_{t+1}; s_0(\nu_{t+1}), \ldots, s_K(\nu_{t+1}) \right] \mid \mathcal{N}_{t+1}) \Bigg\} \Bigg\}. \tag{15}$$

It follows from the envelope property that asset demands are given by

$$q_k(\nu_t; t+1) = -\frac{\frac{\partial V(A(\nu_t); \cdot|\cdot)}{\partial s_k(\nu_t)}}{\frac{\partial V(A(\nu_t); \cdot|\cdot)}{\partial A(\nu_t)}}. \tag{16}$$

This is the multi-period counterpart of Perraudin and Sørensen (1989) who use an one-period version of (16) to derive asset demands from an indirect utility function.

4 LIQUIDITY CONSTRAINTS – DUAL APPROACH

Let us define the auxiliary variable $X(\nu_t)$ in terms of the Lagrange multipliers introduced in (21):

$$X(\nu_t) = \pi(\nu_t) + \zeta(\nu_t), \ \forall \nu_t \in N_t, \ \ t = 1, \ldots, T-1; \quad X(\nu_T) = \pi(\nu_T). \tag{17}$$

$\{X(\nu_t) \mid \mathcal{N}_t\}$, admits the standard interpretation as the marginal utility of wealth, a non-negatively valued stochastic process adapted to $\mathcal{N}_t$. Since $X(\nu_t)$ is $\mathcal{N}_t$-adapted, it may be represented by an L-dimensional vector $\boldsymbol{X}$, $\boldsymbol{X} \in \mathbb{R}_+^L$. Pursuing duality, we will rewrite the expression for lifetime utility in (12) to eliminate $b(\cdot)$, and $q(\cdot)$. This amounts to exploring the saddle point property of the Lagrangean to rewrite Problem 1 as a minimization problem.

By definition (17) above and in view of Proposition 1, $\boldsymbol{X}, \boldsymbol{X} \in \mathbb{R}_+^L$, is an $\mathcal{N}_t$-adapted positive stochastic process that satisfies:

$$X(\nu_t) \geq \frac{1+\sigma_{t+1}}{1+\rho} \sum_{\nu_{t+1} \in N_{t+1} \cap \nu_t} p(\nu_{t+1} \mid \nu_t) X(\nu_{t+1}), \forall \nu_t \in N_t, t = 0, 1, \ldots, T-1. \tag{18}$$

and

$$(1+\sigma_{t+1}) \sum_{\nu_{t+1} \in N_{t+1} \cap \nu_t} p(\nu_{t+1} \mid \nu_t) X(\nu_{t+1}) =$$

$$\sum_{\nu_{t+1}\in N_{t+1}\cap\nu_t} p(\nu_{t+1} \mid \nu_t) X(\nu_{t+1}) \frac{s_j(\nu_{t+1})}{s_j(\nu_t)}, \ \forall j \in \{1,\ldots,K\} \ \forall \nu_t \in N_t, \ t = 0,1,\ldots,T-1. \quad (19)$$

Let $\mathcal{D}$ denote the set of X's that satisfy (18)– (19), a closed convex set in $\mathbb{R}^L_+$. Equality in (18) means that the corresponding liquidity constraint (11) holds as a strict inequality (constraint not tight) and the individual is not constrained in period t. This is simply a restatement of (7). Strict inequality in (18) means that the individual is constrained in period t. Condition (18) says that the marginal utility of wealth is a supermartingale.[9] Conditions (19) require that the gross rates of return, weighted by the respective marginal utility of expenditure, are equalized across all assets which are held in the portfolio in a particular period.

We define the Legendre-Fenchel transform of $v_t(b; \cdot \mid \mathcal{N}_t)$, $\varpi_t(X; \cdot \mid \mathcal{N}_t)$, which is known in the economics literature[10] as the *profit function:*

$$\varpi_t(X(\nu_t); W(\nu_t) \mid \mathcal{N}_t) \ \equiv \ \max_{b(\nu_t)>0} : \ v_t(b(\nu_t); W(\nu_t) \mid \mathcal{N}_t) \ - \ X(\nu_t) b(\nu_t). \qquad (20)$$

This definition originates in the firm's problem, where it applies literally: a profit function $\varpi_t(K_t, p_t)$ gives the maximum profit to be earned when capital K_t is given, and the optimal quantities of all of the firm's variable inputs are employed. The neoclassical theory of the firm treats capital as a quasi-fixed factor, whose adjustment via investment I_t is subject to transactions costs, $c(I_t, K_t)$. The resulting net profit function $\varpi_t(K_t, p_t) - c(I_t, K_t)$ along with the adjoined constraint take the place of the indirect utility function.

Under the assumption that the indirect utility function v_t is concave with respect to b, then the profit function $\varpi_t(X; W \mid \cdot)$ is convex and decreasing in X. Duality, as expressed in Proposition 1 and definitions (17) and (20), leads to the following Proposition.

Proposition 2.

(a) The dual of Problem 1 may be stated as:

$$\min_{\boldsymbol{X}\in\mathcal{D}} : \ E_0 \left\{ \sum_{t=0}^{T} \sum_{\nu_t \in N_t} (1+\rho)^{-t} \left[\varpi_t(X(\nu_t); W(\nu_t) \mid \mathcal{N}_t) + X(\nu_t) h(\nu_t) \right] \right\} + X(N_0) F_0. \quad (21)$$

(b) Problem (21) has a unique solution.

(c) If $\boldsymbol{X}^*$ is a solution to Problem (21), then

[9] Browning and Robb (1985) were the first to recognize that Frisch demands are still useful when capital markets are imperfect and the marginal utility of wealth is a supermartingale, rather than a martingale.

[10] See McFadden (1978) for the first use and Browning *et al.* (1985) for a recent one, which our approach is most closely related to.

$$b^*(\nu_t) = f(X^*(\nu_t) \mid \mathcal{N}_t), \tag{22}$$

is a solution to Problem (12), where $f(\cdot \mid \cdot)$[11] *is defined as*:

$$f_t(x \mid \mathcal{N}_t) = \inf_b : \left\{ b \geq 0;\ \frac{\partial \upsilon_t}{\partial b}(b; W \mid \mathcal{N}_t) \leq x \right\}. \tag{23}$$

(d) The optimal expenditure function given by (22) and (23) yields a vector of demand functions for leisure and other goods that coincide with the Frisch demands.

Proof. The strict concavity of the indirect utility function υ_t implies convexity of the profit function υ_t. This fact implies that the statement of Problem according to Proposition 1 is subject to the theory of convex programming. The finite dimensional representation of uncertainty ensures that the liquidity constraints are represented by linear inequalities, so that Problem 1 is equivalently stated as a problem of minimizing a globally convex function subject to linear inequality constraints.

Part (a) above follows as a classical statement of duality. E.g., see Luenberger (1969), Chapter 8. Existence of a solution is guaranteed, provided that a regularity condition[12] is satisfied. Such a condition is indeed satisfied here. A solution, which suggests that the individual spend her labor income and neither borrow nor invest, satisfies the liquidity constraints and thus is feasible. Financial assets at every period in time represent the current value of initial assets. Under the assumption of strict concavity of υ_t the solution to the dual problem involves minimization of a strictly convex function and is subject to a convex constraint and thus is unique. Thus part (b) follows.

Part (c) follows trivially from the definition of the profit function in (20). For part (d), it suffices to recognize that according to its definition in (20), the profit function $\varpi_t(x; W)$ coincides with the profit function defined by McFadden (1978) for production settings and adapted to systems of preferences by Browning, Deaton and Irish (1985).

□ □

An important economic interpretation of the dual formulation follows from (21) and complements the one obtained by Browning *et al.* (1985). $X(\nu_t)$ denotes the marginal utility of period t expenditure. This follows from (13) and (17). The minimand in (21) denotes discounted lifetime profit from consumption and from the sale of the endowment of labor. To the primal problem of maximizing lifetime expected utility (Proposition 1) there corresponds the dual problem of finding values for marginal utility in every period and state (that is,

[11] f is the inverse function of $\frac{\partial \upsilon(b;W)}{\partial b}$ with respect to b.

[12] See Luenberger (1969), Chapters 8 and 9. The more recent literature, e.g. Duffie (1992), refers to such a condition as Slater condition, *ibid.* p.10, 228.

for the inverse of the price of utility) so that the total expected profit from consumption be minimized. Clearly, Propositions 2 and 3 imply an one-to-one correspondence between $A(\nu_t)$ and $X(\nu_t)$.

We may obtain an interesting corollary of Proposition 2 by multiplying each of the inequalities (11) by $X(\nu_t) - (1+\rho)^{-1} \sum_{\nu_{t+1} \in N_{t+1} \cap \nu_t} p(\nu_{t+1} \mid \nu_t) X(\nu_{t+1}) \frac{s_k(\nu_{t+1})}{s_k(\nu_t)}$, $\forall \nu_t \in N_t,\ t = 0, 1, \ldots, T-1$. We note that each of these terms is positive when (11) holds as an inequality and is zero otherwise. By taking expectations as of $t = 0$, summing up over all the ν_t's, and in view of (19) we have that:

$$X^*(N_0)F_0 + E_0 \left\{ \sum_{t=0}^{T} X^*(\nu_t) h(\nu_t) \right\} = E_0 \left\{ \sum_{t=0}^{T} X^*(\nu_t) b^*(\nu_t) \right\}. \tag{24}$$

Equation (24) constitutes the transformation of the sequence of constraints (11) into a single one resembling (5). $p(\nu_t)X^*(\nu_t)$ is an *individual-specific implicit* system of Arrow-Debreu prices (or shadow prices). If the $p(.)X(.)^*$'s were to be interpreted as Arrow-Debreu state prices and the individual were allowed to sell her labor at these prices at time $t = 0$, then the individual's optimal consumption decisions would be identical to those of the original problem with liquidity constraints.[13] Duality implies that a constrained optimum, such as point C, Figure 1, may be visualized as an unconstrained one.

The interpretation of the solution to the liquidity constrained problem in terms of an equivalent unconstrained problem is seemingly akin to the permanent income hypothesis. The present value of the lifetime endowment of leisure looks like a linear function of the endowment of leisure in each period, but it is not. The respective discount factors, which are the coefficients that multiply each of the terms in the sequence, and which we have interpreted as individualized implicit Arrow-Debreu prices, are themselves functions of initial assets, of individual characteristics that enter as preference parameters, and of the entire specification of the stochastic structure. Those same coefficients should also be used in discounting expenditures.

4.1 Dual Dynamic Programming

We explore the recursiveness properties of the solution to Problem 1 as characterized by Proposition 2 to develop a dual dynamic programming formulation, which in view of Proposition 2 part (d) is interesting in its own right. Our approach adapts to discrete-time settings the method developed by He and Pages (1993). To our knowledge, this is the first such development.

[13] *c.f.* He and Pagès (1993), p. 22.

The dual problem, defined in (21), may be amenable to a dynamic programming formulation by means of a value function in the usual way [Bertsekas (1987)]. Let $\boldsymbol{X} \in \mathcal{D}$, where the set $\mathcal{D} \subset \mathbb{R}_+^L$ was defined so as to satisfy (18) and (19) above. The dual counterpart of the value function defined in (15) is:

$$J_T(X(\nu_{T-1})) = \inf_{\{X(\nu_T)\in\mathcal{D}_T[X(\nu_{T-1}]\}} : E\left\{\varpi_T(X(\nu_T); W(\nu_T) \mid \mathcal{N}_T) \right.$$
$$\left. + X(\nu_T)h(\nu_T) \mid \mathcal{N}_{T-1}\right\}, \forall \nu_{T-1} \in N_{T-1}$$

$$J_{T-1}(X(\nu_{T-2})) = \inf_{\{X(\nu_{T-1})\in\mathcal{D}_{T-1}[X(\nu_{T-2})]\}} : E\left\{\varpi_{T-1}(X(\nu_{T-1}); W(\nu_{T-1}) \mid \mathcal{N}_{T-1}) \right.$$
$$\left. + X(\nu_{T-1})h(\nu_{T-1}) + (1+\rho)^{-1} J_T(X(\nu_{T-1})) \mid \mathcal{N}_{T-2}\right\}, \forall \nu_{T-2} \in N_{T-2}$$

$$J_t(X(\nu_{t-1})) = \inf_{\{X(\nu_t)\in\mathcal{D}_t[X(\nu_{t-1})]\}} : E\left\{\varpi_t(X(\nu_t); W(\nu_t) \mid \mathcal{N}_t) \right.$$
$$\left. + X(\nu_t)h(\nu_t) + (1+\rho)^{-1} J_{t+1}(X(\nu_t))\right\} \tag{25}$$

$$J_0(X(N_0)) = \varpi_0(X(N_0); W(N_0) \mid N_0)) + X(N_0)h(N_0) + (1+\rho)^{-1} J_1(X(N_0));$$

$$J^*(F_0) = \inf_{\{X(N_0)\geq 0\}} : X(N_0)F_0 + J_0(X(N_0)),$$

where we have suppressed dependence on the $s_k(\cdot)$'s and the sets $\mathcal{D}_t[\cdot],\ t = 0, 1, \ldots, T$ are defined as follows:

$$\mathcal{D}_t(X(\nu_{t-1})) = \{X(\nu_{t+1}) \geq 0 : X(\nu_{t+1}) \text{ satisfies (18) and (19)}, \quad \forall \nu_t \in N_t, \forall \nu_{t+1} \in N_{t+1} \cap \nu_t\}. \tag{26}$$

Clearly, $\mathcal{D}_t[\cdot]$ is a closed convex set in $\mathbb{R}_+^{n_t}$, where $n_t = |N_t \cap \nu_{t-1}|$, and thus a convex set.

This reformulation of the stochastic dynamic programming problem would be standard, were it not for constraining the unknown function $X(\nu_t)$ at each state to satisfy a constraint in terms of expectations. At each state in period $T-1$, the individual decides on $X(\nu_T)$ conditional on $X(\nu_{T-1})$. Similarly, at each state in period $T-2$, the individual decides on $X(\nu_{T-1})$ conditional on $X(\nu_{T-2})$, and so on back to period 0, when $X(N_0)$ is chosen. Once $X(N_0)$ has been chosen, then all contingent plans may carried out as uncertainty unfolds over time. We note that if an individual is constrained in some state ν_t then $X(\nu_{t+1})$ depends only on ν_{t+1} and not on $X(\nu_t)$; otherwise it does. Therefore, existence of a sequence of consecutive states $\{\nu_{\underline{t}}, \ldots, \nu_{\bar{t}}\}$ during which the individual is unconstrained implies that $X(\nu_{\bar{t}})$ depends only upon $X(\nu_{\underline{t}})$. The primal version of this result is, of course, well-known. We now put our results formally in terms of Proposition 3.

Proposition 3.

(a) There exists a process $X_c(\nu_t)$, which is adapted to $\mathcal{N}_t$ and is obtained by solving (30) recursively as explained below, such that $\boldsymbol{X}^*$, the optimal solution to Problem 1 may be characterized as follows:

1. *If $X^*(\nu_t) \leq X_c(\nu_t)$, then $X^*(\nu_t)$ satisfies (18) as an equality.*

2. *If $X^*(\nu_t) = X_c(\nu_t)$, then $X^*(\nu_t)$ satisfies (18) as an inequality.*

We say that 1 above is the unconstrained case, with (11) holding as an inequality , and that 2 is the constrained case, with (11) holding as an equality.

(b) The threshold values $\{X_c(\nu_t) \mid \mathcal{N}_t\}$ are defined recursively according to (30), for $t = T, T-1, \ldots, 0$ and form a $\mathcal{N}_t$-measurable process that characterize fully the optimal policy.

(c) Duality implies that the quantity in (31) below, evaluated at the optimum, is equal to minus the total value of financial assets as of the beginning of period t, $-F^*(\nu_t) = -A(\nu_t) + h(\nu_t)$.

(d) To the threshold values $\{X_c(\nu_t) \mid \mathcal{N}_t\}$ there correspond threshold values $A_c(\nu_t)$ for $A(\nu_t)$, the state variable of the primal problem, such that $b^*(\nu_t) = A(\nu_t)$, if $A(\nu_t) \leq A_c(\nu_t)$, and $b(\nu_t) < A(\nu_t)$, otherwise, where $A(\nu_t)$ is defined in (10).

(e) The envelope property holds, that is:

$$\frac{d}{dF_0} J^*(F_0) = X(N_0) = \frac{\partial}{\partial b(N_0)} v_0(b(N_0); \cdot \mid N_0), \tag{27}$$

and similarly for $0 < t \leq T$.

Proof: We note first that the minimization problem in the right hand side of (25) for $t = T$ involves a minimand that is a convex function of $X(\nu_T)$ and is subject to $X(\nu_T) \in \mathcal{D}_t[X(\nu_{T-1})]$, which is a closed convex set in $\mathbb{R}_+^{n_T}$, where $n_T = |N_T \cap \nu_{T-1}|$. We describe the solution by defining $X_c(\nu_T)$ as the solution to:

$$\frac{\partial}{\partial X(\nu_T)}\varpi_T(\cdot \mid \mathcal{N}_T) + h(\nu_T) = 0, \;\; \forall \nu_T \in N_T \cap \nu_{T-1}. \tag{28}$$

It follows from the properties of ϖ_T that if a solution to the above equation exists, it is unique under our assumptions. The minimand in the right hand side of (25) for $t = T$ and the constraint that $X(\nu_T) \in \mathcal{D}_T(X_{\nu_{T-1}})$ both involve expectations with respect to ν_T, conditional on ν_{T-1} and subject to $X(\nu_{T-1})$. Therefore, the optimality condition for period T may be interpreted as $b(\nu_T) = h(\nu_T) + \xi(X(\nu_{T-1}))$, where $\xi(.)$, $\xi(.) \geq 0$, denotes the Lagrange multiplier corresponding to constraint (18). If the latter at the optimum holds as equality, that is

$$X^*(\nu_{T-1}) = \frac{1+\sigma_T}{1+\rho} \sum_{\nu_T \in N_T \cap \nu_{T-1}} p(\nu_T \mid \nu_{T-1}) X^*(\nu_T), \forall \nu_{T-1} \in N_{T-1}, \tag{29}$$

then the individual is unconstrained in period $T-1$, in which case $\xi(X(\nu_{T-1})) > 0$, and $X^*(\nu_T) < X_c(\nu_T)$. In that case $X(\nu_T)$ is a function of $X(\nu_{T-1})$. If, on the other hand, (18) holds as inequality, the individual is constrained in period $T-1$. In such a case $b(\nu_T) = h(\nu_T)$ and $X^*(\nu_T) = X_c(\nu_T)$, and thus independent of $X(\nu_{T-1})$.

Let us define $X_c(\nu_{T-1})$ as the unique value of x which satisfies:

$$\frac{\partial \varpi_{T-1}}{\partial x}(x; W(\nu_{T-1}) \mid \mathcal{N}_{T-1}) + h(\nu_{T-1}) + \frac{1}{1+\rho}\frac{\partial}{\partial x} J_T(x) = 0, \tag{30}$$

By the envelope property, $\frac{\partial}{\partial x} J_T(x) = -\xi(x)$. It follows by comparing (28) and (30) that *cet. par.* X_c is nonincreasing over time.

Working recursively we may establish that since ϖ_t is convex and decreasing in $X(\nu_t)$, so is $J_{t+1}(X(\nu_t))$, the dual value function defined in (25). Should the quantity

$$\frac{\partial \varpi_t}{\partial X}(X; W(\nu_t) \mid \mathcal{N}_t) + h(\nu_t) + \frac{1}{1+\rho}\frac{\partial}{\partial X} J_{t+1}(X) \tag{31}$$

become 0 in an interior point of $\mathcal{D}_t[X(\nu_{t-1})]$, then such a point is unique and defines the optimal value $X^*(\nu_t)$. Otherwise, the minimum is characterized by a point on the boundary of $\mathcal{D}_t[X(\nu_{t-1})]$, such that expression (31) is negative, and (18) holds as an equality.[14] The optimal value of $X(\nu_t)$ may thus be characterized as follows:

$$X^*(\nu_t) = \max\left\{ X_c(\nu_t), \frac{1+\sigma_{t+1}}{1+\rho} \sum_{\nu_{t+1} \in N_{t+1} \cap \nu_t} p(\nu_{t+1} \mid \nu_t) X^*(\nu_{t+1}) \right\}. \tag{32}$$

[14]It is worth noting that constrained (unconstrained) behavior corresponds to an optimum in the interior (boundary).

The threshold value X_c defines a critical boundary in the nonnegative halfspace of $\mathbb{R}_+^{n_t}$, where $n_t = |N_t \cap \nu_{t-1}|$. This boundary splits the dual halfspace into two regions, the lower one corresponding to unconstrained behavior and the higher one to constrained behavior [Figure 2]. Returning to the primal space, this implies a separation in the space of the market value of wealth in the beginning of each period, which includes financial assets plus the value of the endowment of leisure.

Finally, to show uniqueness of the optimal solution, it suffices to show that (30) implies a unique $X(N_0)$. Since ϖ_t is a convex decreasing function and $J_1(X(N_0))$ is a convex function of $X(N_0)$ then generically a unique non-negative $X(N_0)$ exists for which the infimum is attained. Parts (a) and (b) have been proven.

Part (c) follows as a standard statement of the envelope property.[15] It allows an interpretation of the condition for optimality. If $-A(\nu_t) + h(\nu_t) = 0$, the individual is constrained in period $t-1$; if, on the other hand, $-A(\nu_t) + h(\nu_t) \leq 0$, then the individual is unconstrained in period $t-1$.

To establish part (d), we note from Proposition 2, part (c), that the existence of $X_c(\nu_t)$ implies the existence of $b_c(\nu_t)$, $b_c(\nu_t) = f(X_c(\nu_t) \mid \mathcal{N}_t)$ such that if $X(\nu_t) = X_c(\nu_t)$, then $A_c(\nu_t) \equiv b_c(\nu_t) = b^*(\nu_t)$. It then follows from the monotonicity and strict concavity of v with respect to b that:

$$b^*(\nu_t) = A(\nu_t), \quad \text{if} \quad A(\nu_t) \leq A_c(\nu_t) \equiv f(X_c(\nu_t)|\mathcal{N}_t);$$

$$b^*(\nu_t) < A(\nu_t), \quad \text{if} \quad A(\nu_t) > A_c(\nu_t).$$

Finally part (e) for $t = 0$ follows from the definition of $J^*(F_0)$ by differentiating with respect to F_0. For $t > 0$ we note that the problem may be truncated at any point in time t, $t < T$, and state ν_t and the value of remaining utility be considered as a function of financial wealth as of the beginning of period t and conditional on state ν_t, $F(\nu_t) = \sum_{k=0}^{K} q_k(\nu_{t-1}; t) s_k(\nu_t)$. Then everything applies with the obvious change in notation. The proof of Proposition 3 is thus complete. □ □

The advantage of duality theory readily follows from (32). The multidimensional primal problem is tranformed into a scalar dual problem, where the unknown dual variable follows a linear stochastic difference equation, subject to a boundary. The boundary is defined in terms of fundamentals and defines the critical region.

[15] This is one of the ways in which the discrete- and continuous-time formulations differ. When trading takes place continuously, there is no counterpart to before and after trading. The respective proposition in He and Pagès gives an expression for wealth.

4.2 Consequences for Econometric Modelling

In econometric research where the underlying analytical model may be formulated as a stochastic dynamic programming model, it is testable implications of (7) or (18) that have been relied upon principally.[16] Most of the literature does not specify in detail the sources of stochastic shocks. It is, therefore, essential that the analytical premises of such empirical research may be as general as possible.

It may be argued that our approach is more general than previous ones, in the sense that our dual dynamic programming formulation is set within a very general representation of uncertainty. Generalizing the assumption of finite dimensionality of that representation is a matter of elegance. It is thus quite important that both b^* and X^* are in general $\mathcal{N}_t$-adapted. Consequently, they vary continuously with the underlying parameters of the decision problem.

A stronger statement may be made about the dependence of the Lagrange multiplier on total wealth. It is easiest to think about this as of the beginning of the process. We note that from the definition of $J^*(F_0)$ and Proposition 3 we may obtain that $X(N_0)$ is a monotone decreasing function of $A_0 \equiv F_0 + h(N_0)$. This is straightforward once we recall that $\varpi_0(\cdot)$ and $J_1(\cdot)$ are non-increasing convex functions of $X(N_0)$. By working in like manner we can show the same for $X(\nu_t)$. As we discussed earlier above, if the $X(\nu_t)'$s were known as functions of parameters and prices then the problem could be solved as if (24) were the only constraint. That same fact suggests in estimation one should be careful so that the most general dependence for X on the information set is allowed for.

No general statement about the threshold value of assets may be made, except that it, too, is $\mathcal{N}_t$-adapted. About the determinants of Frisch demands we note that even though their functional form is invariant to whether or not constraints are active, their reduced forms in terms of the determinants of the Lagrange multipliers are not. It is this fact which Hajivassiliou and Ioannides (1990) exploit in developing a switching – regression estimation model.

We indicated above that the existence of a sequence of unconstrained states $\{\nu_{\underline{t}}, \ldots, \nu_{\bar{t}}\}$ implies that $X(\nu_{\bar{t}})$ depends only upon $X(\nu_{\underline{t}})$, and marginal utility of wealth is, from (18), a martingale in the intervening years. Similarly, if an individual is constrained in some state ν_t then $X(\nu_{t+1})$ depend only on ν_{t+1} and not on $X(\nu_t)$. These implications are, of course, in principle, testable via the overidentifying restrictions implied by the martingale hypothesis.

Frisch demand theory exploits a correspondence between marginal utility of wealth and wealth for a given individual. The role of the Lagrange multiplier as a sufficient statistic in the

[16]See MaCurdy (1985) for a review of empirical research using Frisch demand models.

life cycle model may be equivalently assumed by total assets (or, net asset decumulation). Unfortunately, many empirical studies of individuals' behavior are forced to rely on data (such as the U.S. Panel Study of Income Dynamics) where neither assets nor net asset decumulation are observed, but still retain Frisch demand theory in structuring empirical investigations.

Firms' behavior lends itself neatly to a demonstration of the usefulness of duality theory in empirical investigations. Unlike the case of individuals, the objective which publicly traded firms are supposed to optimize according to neoclassical theory may be observable. The dual variable of interest in the case of firms' behavior is associated with a capital accumulation constraint and thus interpreted as the marginal value of capital, known as *marginal* q. Under certain identifying assumptions [Hayashi (1982)] *marginal* q may be equated to *average* q, and thus observable as the ratio of the value of a firm's capital, which is provided by the stock market, to the replacement cost of capital. Alternatively, it may be computed from the present value of the future stream of net profits associated with a marginal unit of investment. Such a linkage with the $q-$theory of investment enhances the attractiveness of duality theory. If investment is unconstrained, duality leads to a stochastic difference equation which is linear in the unknown (dual) variable.

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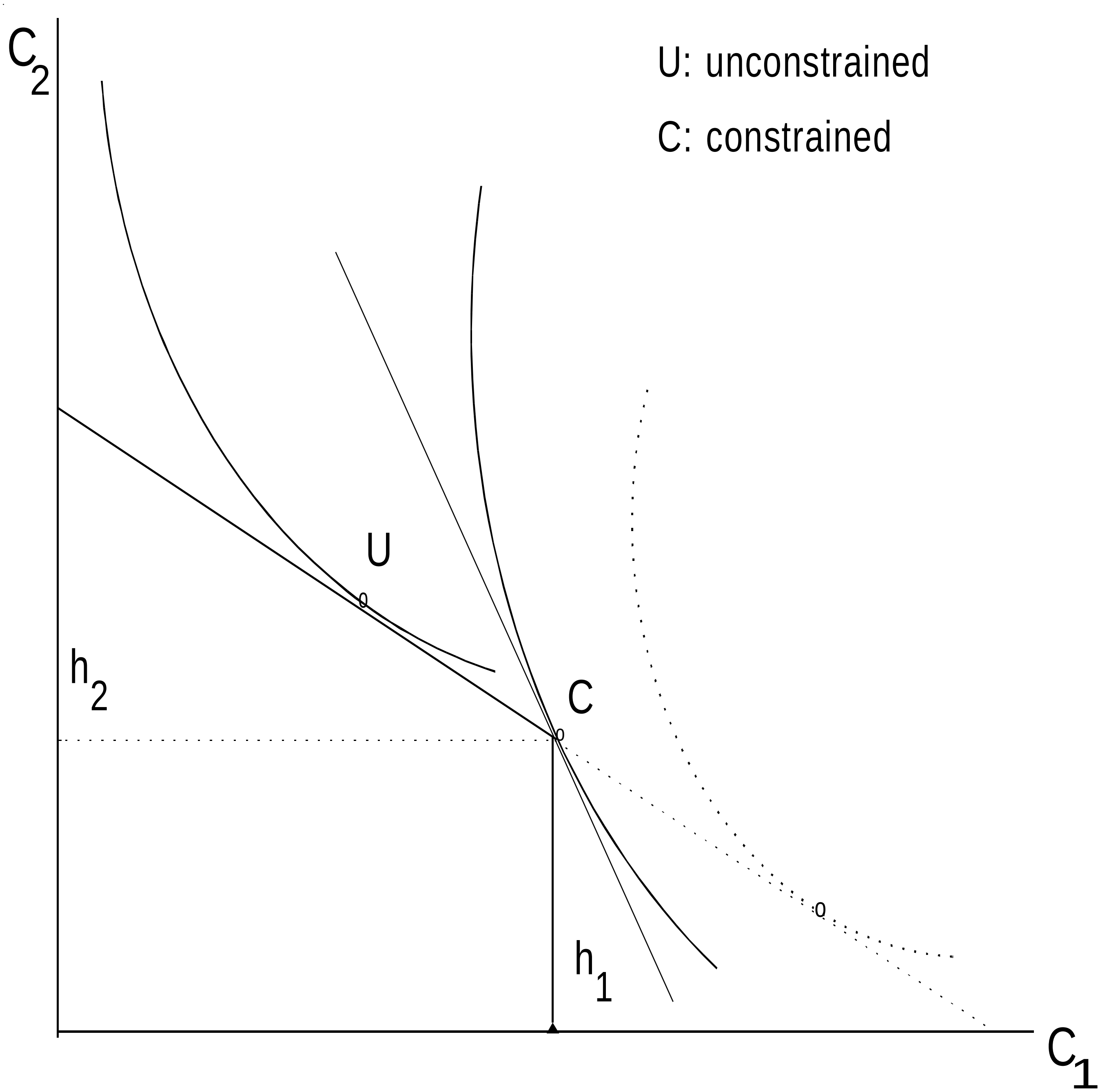

Figure 1

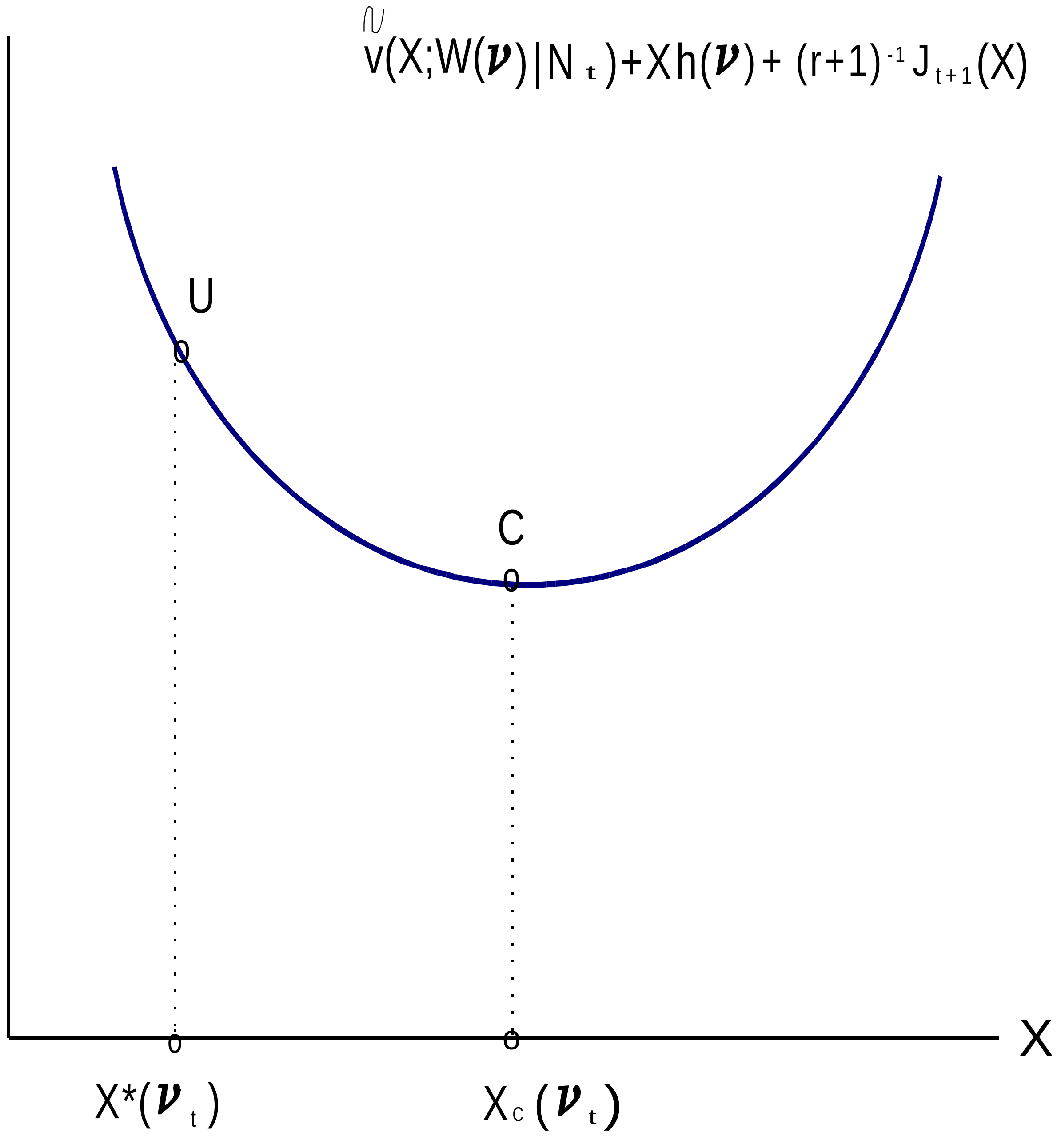

Figure 2