Stability, Multiplicity, and Sun Spots (Blanchard-Kahn conditions)

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Introduction

- What do we mean with non-unique solutions?
    - multiple solution versus multiple steady states
- What are sun spots?
- Are models with sun spots scientific?

Terminology

- Definitions are very clear
  - (use in practice can be sloppy)

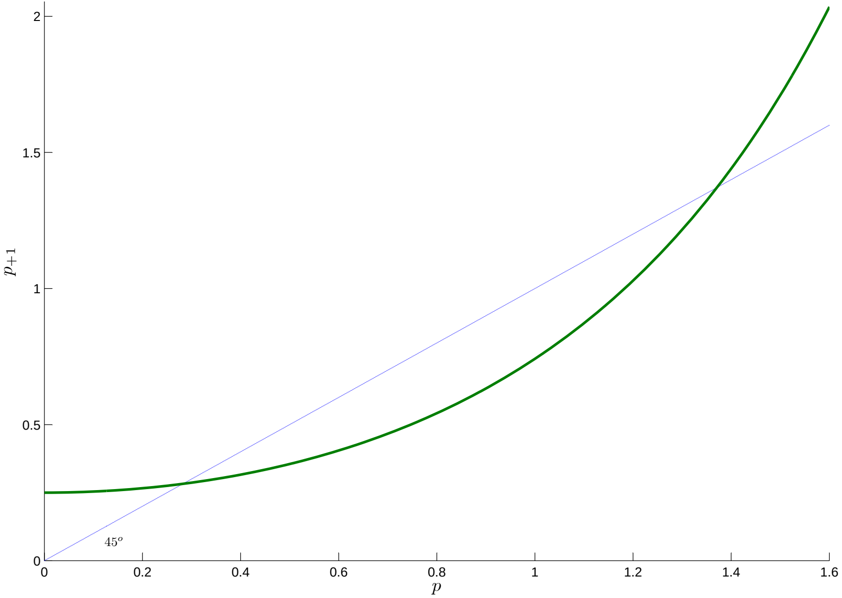
Model:

$$H(p_{+1}, p) = 0$$

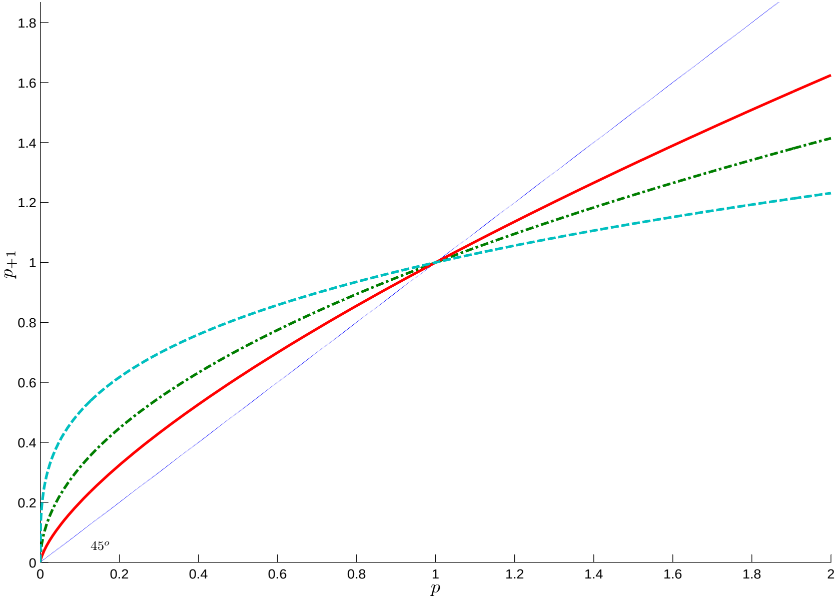
Solution:

$$p_{+1} = f(p)$$

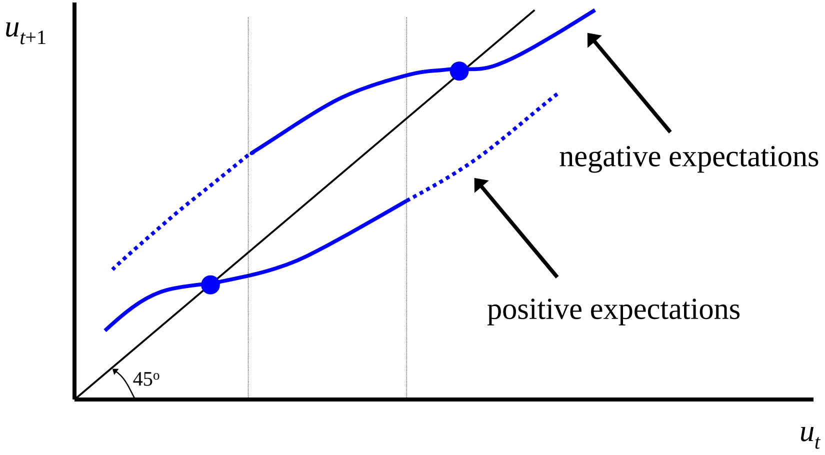
Unique solution & multiple steady states



Multiple solutions & unique (non-zero) steady state

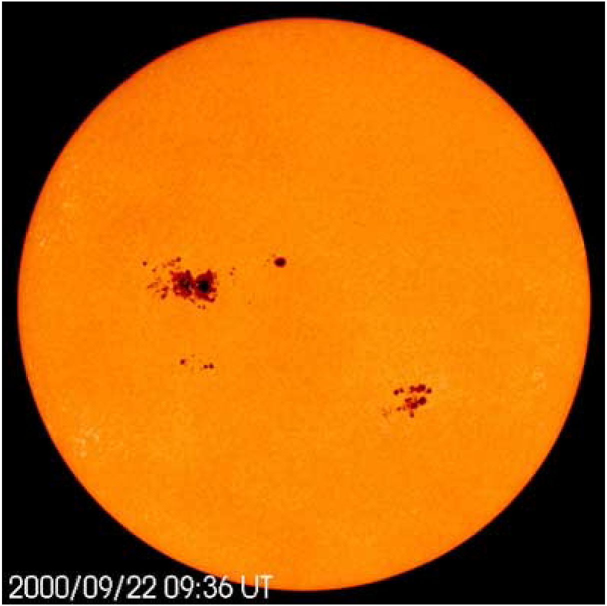


Multiple steady states & sometimes multiple solutions

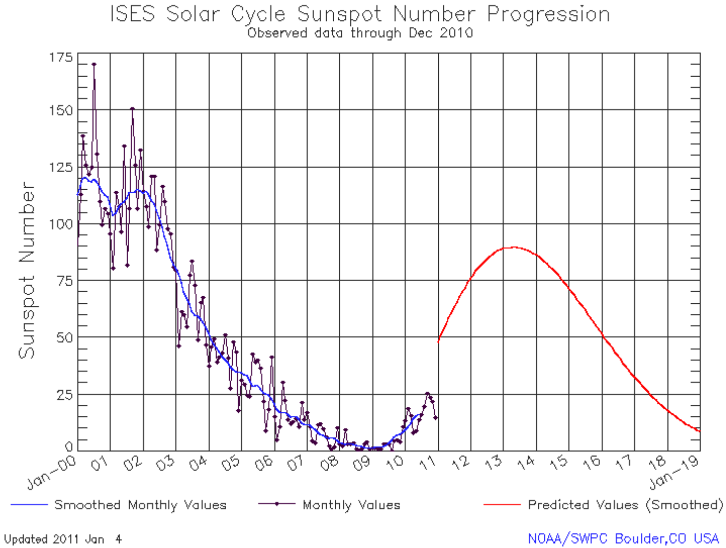


From Den Haan (2007)

Large sun spots (around 2000 at the peak)



Sun spot cycle (almost at peak again)



Sun spots in economics

- **Definition:** a solution is a sun spot solution if it depends on a stochastic variable *from outside system*
- **Model:**

$$
\begin{aligned}
0 &= \mathsf{E}H(p_{t+1}, p_t, d_{t+1}, d_t) \\
d_t &: \text{exogenous random variable}
\end{aligned}
$$

Sun spots in economics

- **Non-sun-spot solution:**

$$p_t = f(p_{t-1}, p_{t-2}, \cdots, d_t, d_{t-1}, \cdots)$$

- **Sun spot:**

$$\begin{aligned} p_t &= f(p_{t-1}, p_{t-2}, \cdots, d_t, d_{t-1}, \cdots, s_t) \\ s_t &: \text{ random variable with } \mathrm{E}[s_{t+1}] = 0 \end{aligned}$$

Sun spots and science

Why are sun spots attractive

- sun spots: s_t matters, just because agents believe this
  - self-fulfilling expectations don't seem that unreasonable
- sun spots provide many sources of shocks
  - number of sizable fundamental shocks small

Sun spots and science

Why are sun spots not so attractive

- Purpose of science is to come up with predictions
  - If there is one sun spot solution, there are zillion others as well
- Support for the conditions that make them happen not overwhelming
  - you need sufficiently large increasing returns to scale or externality

Overview

1. Getting started
   - simple examples
2. General derivation of Blanchard-Kahn solution
   - When unique solution?
   - When multiple solution?
   - When no (stable) solution?
3. When do sun spots occur?
4. Numerical algorithms and sun spots

Getting started

-

$$\textbf{Model:} \quad y_t = \rho y_{t-1}$$

Getting started

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- infinite number of solutions, independent of the value of ρ

Getting started

- **Model:** $$\begin{array}{l} y_{t+1} = \rho y_t \\ y_0 \text{ is given} \end{array}$$

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- unique solution, independent of the value of ρ

Getting started

- Blanchard-Kahn conditions apply to models that add as a requirement that the series do not explode

Model:
$$y_{t+1} = \rho y_t$$
$$y_t \text{ cannot explode}$$

- $\rho > 1$: nique solution, namely $y_t = 0$ for all t
- $\rho < 1$: many solutions
- $\rho = 1$: many solutions
  - be careful with $\rho = 1$, uncertainty matters

State-space representation

$$Ay_{t+1} + By_t = \varepsilon_{t+1}$$

$$\mathsf{E}\left[\varepsilon_{t+1} | I_t\right] = 0$$

y_t : is an $n \times 1$ vector
$m \leq n$ elements are not determined

some elements of ε_{t+1} are not exogenous shocks but prediction errors

Neoclassical growth model and state space representation

$$\left(\exp(z_t)k_{t-1}^{\alpha} + (1-\delta)k_{t-1} - k_t\right)^{-\gamma} =$$
$$\mathsf{E}\left[\left.\begin{array}{c} \beta\left(\exp(z_{t+1})k_t^{\alpha} + (1-\delta)k_t - k_{t+1}\right)^{-\gamma} \\ \times\left(\alpha\exp\left(z_{t+1}\right)k_t^{\alpha-1} + 1 - \delta\right) \end{array}\right| I_t\right]$$

or equivalently without $\mathsf{E}[\cdot]$

$$\begin{gathered}\left(\exp(z_t)k_{t-1}^{\alpha} + (1-\delta)k_{t-1} - k_t\right)^{-\gamma} = \\ \beta\left(\exp(z_{t+1})k_t^{\alpha} + (1-\delta)k_t - k_{t+1}\right)^{-\gamma} \\ \times\left(\alpha\exp\left(z_{t+1}\right)k_t^{\alpha-1} + 1 - \delta\right) \\ + e_{\mathsf{E},t+1}\end{gathered}$$

Neoclassical growth model and state space representation

Linearized model:

$$k_{t+1} = a_1 k_t + a_2 k_{t-1} + a_3 z_{t+1} + a_4 z_t + e_{\mathsf{E},t+1}$$

$$z_{t+1} = \rho z_t + e_{z,t+1}$$

$$k_0 \text{ is given}$$

- k_t is end-of-period t capital
  - $\Longrightarrow k_t$ is chosen in t

Neoclassical growth model and state space representation

$$\begin{bmatrix} 1 & 0 & a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_{t+1} \\ k_t \\ z_{t+1} \end{bmatrix} + \begin{bmatrix} a_1 & a_2 & a_4 \\ -1 & 0 & 0 \\ 0 & 0 & -\rho \end{bmatrix} \begin{bmatrix} k_t \\ k_{t-1} \\ z_t \end{bmatrix} = \begin{bmatrix} e_{\mathsf{E},t+1} \\ 0 \\ e_{z,t+1} \end{bmatrix}$$

Dynamics of the state-space system

$$Ay_{t+1} + By_t = \varepsilon_{t+1}$$

$$\begin{aligned} y_{t+1} &= -A^{-1}By_t + A^{-1}\varepsilon_{t+1} \\ &= Dy_t + A^{-1}\varepsilon_{t+1} \end{aligned}$$

Thus

$$y_{t+1} = D^t y_1 + \sum_{l=1}^{t+1} D^{t+1-l} A^{-1} \varepsilon_l$$

Jordan matrix decomposition

$$D = P\Lambda P^{-1}$$

- Λ is a diagonal matrix with the eigen values of D
- without loss of generality assume that $|\lambda_1| \geq |\lambda_2| \geq \cdots |\lambda_n|$

Let

$$P^{-1} = \begin{bmatrix} \tilde{p}_1 \\ \vdots \\ \tilde{p}_n \end{bmatrix}$$

where $\tilde{p}_i$ is a $(1 \times n)$ vector

Dynamics of the state-space system

$$
\begin{aligned}
y_{t+1} &= D^t y_1 + \sum_{l=1}^{t+1} D^{t+1-l} A^{-1} \varepsilon_l \\
&= P \Lambda^t P^{-1} y_1 + \sum_{l=1}^{t+1} P \Lambda^{t+1-l} P^{-1} A^{-1} \varepsilon_l
\end{aligned}
$$

Dynamics of the state-space system

multiplying dynamic state-space system with P^{-1} gives

$$P^{-1}y_{t+1} = \Lambda^t P^{-1} y_1 + \sum_{l=1}^{t+1} \Lambda^{t+1-l} P^{-1} A^{-1} \varepsilon_l$$

or

$$\tilde{p}_i y_{t+1} = \lambda_i^t \tilde{p}_i y_1 + \sum_{l=1}^{t+1} \lambda_i^{t+1-l} \tilde{p}_i A^{-1} \varepsilon_l$$

recall that y_t is $n \times 1$ and $\tilde{p}_i$ is $1 \times n$. Thus, $\tilde{p}_i y_t$ is a scalar

Model

1. $\tilde{p}_i y_{t+1} = \lambda_i^t \tilde{p}_i y_1 + \sum_{l=1}^{t+1} \lambda_i^{t+1-l} \tilde{p}_i A^{-1} \varepsilon_l$
2. $\mathsf{E}[\varepsilon_{t+1} | I_t] = 0$
3. m elements of y_1 are not determined
4. y_t cannot explode

Reasons for multiplicity

1. There are free elements in y_1
2. The only constraint on $e_{\mathsf{E},t+1}$ is that it is a prediction error.
   - This leaves lots of freedom

Eigen values and multiplicity

- Suppose that $|\lambda_1| > 1$
- To avoid explosive behavior it *must* be the case that

1. $\tilde{p}_1 y_1 = 0$ and
2. $\tilde{p}_1 A^{-1} \varepsilon_l = 0 \quad \forall l$

How to think about #1?

$$\tilde{p}_1 y_1 = 0$$

- Simply an additional equation to pin down some of the free elements
- Much better: This is the policy rule in the first period

How to think about #1?

$$\tilde{p}_1 y_1 = 0$$

Neoclassical growth model:

- $y_1 = [k_1, k_0, z_1]^T$
- $|\lambda_1| > 1$, $|\lambda_2| < 1$, $\lambda_3 = \rho < 1$
- $\tilde{p}_1 y_1$ pins down k_1 as a function of k_0 and z_1
    - this is the policy function in the first period

How to think about #2?

$$\tilde{p}_1 A^{-1} \varepsilon_l = 0 \quad \forall l$$

- This pins down $e_{\mathsf{E},t}$ as a function of $\varepsilon_{z,t}$
- That is, the prediction error must be a function of the structural shock, $\varepsilon_{z,t}$, and cannot be a function of other shocks,
  - i.e., there are no sunspots

How to think about #2?

$$\tilde{p}_1 A^{-1} \varepsilon_l = 0 \quad \forall l$$

Neoclassical growth model:

- $\tilde{p}_1 A^{-1} \varepsilon_t$ says that the prediction error $e_{\mathsf{E},t}$ of period t is a fixed function of the innovation in period t of the exogenous process, $e_{z,t}$

How to think about #1 combined with #2?

$$\tilde{p}_1 y_t = 0 \quad \forall t$$

- Without sun spots
  - i.e. with $\tilde{p}_1 A^{-1} \varepsilon_t = 0 \;\; \forall t$
- k_t is pinned down by k_{t-1} and z_t *in every period*.

Blanchard-Kahn conditions

- Uniqueness: For every free element in y_1, you need one $\lambda_i > 1$
- Multiplicity: Not enough eigenvalues larger than one
- No stable solution: Too many eigenvalues larger than one

How come this is so simple?

- In practice, it is easy to get

$$Ay_{t+1} + By_t = \varepsilon_{t+1}$$

- How about the next step?

$$y_{t+1} = -A^{-1}By_t + A^{-1}\varepsilon_{t+1}$$

- **Bad news**: A is often not invertible
- **Good news**: Same set of results can be derived
  - Schur decomposition (See Klein 2000)

How to check in Dynare

Use the following command after the model & initial conditions part

```
check;
```

Example - x predetermined - 1st order

$$
\begin{aligned}
x_{t-1} &= E_t\left[\phi x_t + z_{t+1}\right] \\
z_t &= 0.9 z_{t-1} + \varepsilon_t
\end{aligned}
$$

- $|\phi| > 1$: Unique stable fixed point
- $|\phi| < 1$: No stable solutions; too many eigenvalues > 1

Example - x predetermined - 2nd order

$$\begin{aligned} \phi_2 x_{t-1} &= E_t\left[\phi_1 x_t + x_{t+1} + z_{t+1}\right] \\ z_t &= 0.9 z_{t-1} + \varepsilon_t \end{aligned}$$

- $\phi_1 = -2.25$, $\phi_2 = -0.5$: Unique stable fixed point
  $(1+\phi_1 L - \phi_2 L^2)x = (1-2L)(1-\frac{1}{4}L)$
- $\phi_1 = -3.5$, $\phi_2 = -3$: No stable solution; too many eigenvalues > 1
  $(1+\phi_1 L - \phi_2 L^2)x = (1-2L)(1-1.5L)$
- $\phi_1 = -1$, $\phi_2 = -0.25$: Multiple stable solutions; too few eigenvalues > 1
  $(1+\phi_1 L - \phi_2 L^2)x = (1-0.5L)(1-0.5L)$

Example - x not predetermined - 1st order

$$\begin{aligned} x_t &= E_t\left[\phi x_{t+1} + z_{t+1}\right] \\ z_t &= 0.9 z_{t-1} + \varepsilon_t \end{aligned}$$

- $|\phi| < 1$: Unique stable fixed point
- $|\phi| > 1$: Multiple stable solutions; too few eigenvalues > 1

References

- Blanchard, Olivier and Charles M. Kahn, 1980,The Solution of Linear Difference Models under Rational Expectations, Econometrica, 1305-1313.
- Den Haan, Wouter J., 2007, Shocks and the Unavoidable Road to Higher Taxes and Higher Unemployment, Review of Economic Dynamics, 348-366.
  - simple model in which the size of the shocks has long-term consequences
- Farmer, Roger, 1993, The Macroeconomics of Self-Fulfilling Prophecies, The MIT Press.
  - textbook by the pioneer
- Klein, Paul, 2000, Using the Generalized Schur form to Solve a Multivariate Linear Rational Expectations Model
  - in case you want to do the analysis without the simplifying assumption that A is invertible