Parameterized Expectations Algorithm

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June 29, 2011

Overview

- Two PEA algorithms
- Explaining stochastic simulations PEA
- Advantages and disadvantages
- Improvements of Maliar, Maliar & Judd
- Extensions
  - learning
  - combining with perturbation

Model

$$
\begin{aligned}
c_t^{-\nu} &= \mathsf{E}_t\left[\beta c_{t+1}^{-\nu}\left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta\right)\right] \\
c_t + k_{t+1} &= z_t k_t^{\alpha} + (1-\delta)\, k_t \\
\ln(z_{t+1}) &= \rho \ln(z_t) + \varepsilon_{t+1} \\
& \varepsilon_{t+1} \sim N(0, \sigma^2) \\
& k_1, z_1 \text{ given}
\end{aligned}
$$

k_t is beginning-of-period t capital stock

Two types of PEA

1. Standard projections algorithm:
    1. parameterize $\mathsf{E}_t\left[\cdot\right]$ with $P_n(k_t, z_t; \eta_n)$
    2. solve c_t from
    $$c_t = \left(P_n(k_t, z_t; \eta_n)\right)^{-1/\nu}$$
    and k_{t+1} from budget constraint
2. Simulations PEA

Stochastic simulations PEA

1. Simulate $\{z_t\}_{t=1}^{T}$
2. Let η_n^1 be initial guess for η_n

Stochastic simulations PEA

3. Iterate until η_n^i converges using following scheme
   1. Generate $\{c_t, k_{t+1}\}_{t=1}^{T}$ using
      $$\begin{aligned} c_t^{-\nu} &= P_n(k_t, z_t; \eta_n^i) \\ k_{t+1} &= z_t k_t^{\alpha} + (1-\delta)\, k_t - c_t \end{aligned}$$
   2. Generate $\{y_{t+1}\}_{t=1}^{T-1}$ using
      $$y_{t+1} = \beta c_{t+1}^{-\nu} \left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right)$$
   3. Let
      $$\hat{\eta}_n^i = \arg\min_{\eta} \sum_{t=T_{\text{begin}}}^{T} \frac{\left(y_{t+1} - P_n(k_t, z_t; \eta)\right)^2}{T}$$
   4. Update upsing
      $$\eta_n^{i+1} = \omega \hat{\eta}_n^i + (1-\omega)\, \eta_n^i \text{ with } 0 < \omega \leq 1$$

Stochastic simulations PEA

- $T_{\text{begin}} >> 1$ (say 500 or 1,000)
  - ensures possible bad period 1 values don't matter
- $\omega < 1$ improves stability

Stochastic simulations PEA

- Idea of regression:

$$y_{t+1} \approx P_n(k_t, z_t; \eta) + u_{t+1},$$

- u_{t+1} is a prediction $\Longrightarrow$ u_{t+1} is orthogonal to regressors
- Suppose

$$P_n(k_t, z_t; \eta) = \exp\left(a_0 + a_1 \ln k_t + a_2 \ln z_t\right).$$

- You are *not* allowed to run the linear regression

$$\ln y_{t+1} = a_0 + a_1 \ln k_t + a_2 \ln z_t + \tilde{u}_{t+1}$$

Why not?

PEA & RE

- Suppose η_n^* is the fixed point we are looking for
- So $P_n(k_t, z_t; \eta_n^*)$ is best predictor of $\mathsf{E}_t\left[\cdot\right]$
- Does this mean that solution is a rational expectations equilibrium?

Disadvantages of stoch. sim. PEA

- The inverse of $X'X$ may be hard to calculate for higher-order approximations
- Regression points are clustered $\Longrightarrow$ low precission
    - recall that even equidistant nodes is not enough for uniform convergence
      "nodes" are even less spread out with simulations PEA)

Disadvantages of stoch. sim. PEA

- Projection step has sampling error
    - this disappears at slow rate (especially with serial correlation)

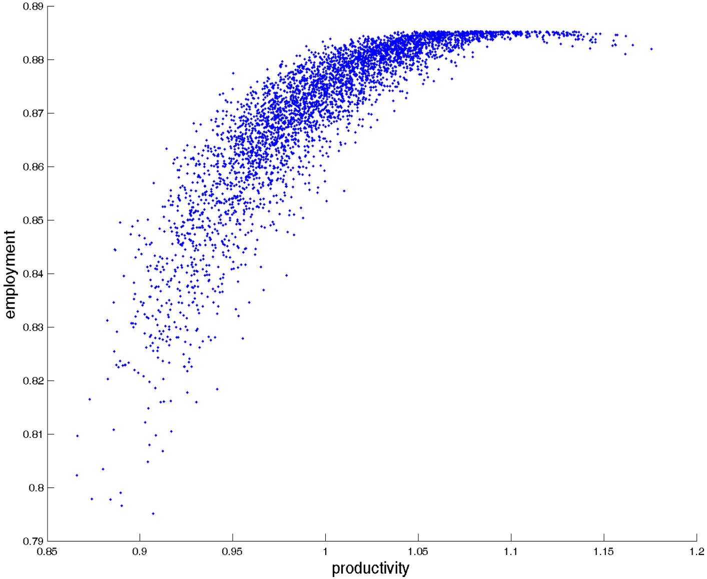
Advantages of stoch. sim. PEA

- Regression points are clustered
  $\Longrightarrow$ better fit *where it matters* **IF** functional form is poor
  (with good functional form it is better to spread out points)

Advantages of stoch. sim. PEA

- Grid: you may include impossible points
  Simulation: model iself tells you which nodes to include
  - (approximation also important and away from fixed point you may still get in weird places of the state space)

Odd shapes ergodic set in matching model



Improvements proposed by Maliar, Maliar & Judd

1. Use flexibility given to you
2. Use $\widehat{\mathrm{E}}\left[y_{t+1}\right]$ instead of y_{t+1} as regressand
   - $\widehat{\mathrm{E}}\left[y_{t+1}\right]$ is numerical approximation of $\mathrm{E}[y_{t+1}]$
   - even with poor approximation the results improve !!!
3. Improve regression step

Use flexibility

1. Many E[]'s to approximate.
    1. Standard approach:

$$c_t^{-\nu} = \mathsf{E}_t \left[\beta c_{t+1}^{-v} \alpha \beta c_{t+1}^{-\nu} \left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right) \right]$$

    2. Alternative:

$$k_{t+1} = \mathsf{E}_t \left[k_{t+1} \beta \alpha \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\nu} \left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right) \right]$$

        - Such transformations can make computations easier *but* can also affect stability of algorithm (for better or worse)

2. $P_n(k, z; \eta)$ could be linear (before or after transformation)

E[y] instead of y as regressor

- $\mathsf{E}[y_{t+1}] = \mathsf{E}[f(\varepsilon_{t+1})]$ with $\varepsilon_{t+1} \sim N(0, \sigma^2)$
  $\Longrightarrow$ Hermite Gaussian quadrature can be used
  (MMJ: using $\widehat{\mathsf{E}}\left[y_{t+1}\right]$ calculated using one node is better than using y_{t+1})
- Key thing to remember: sampling uncertainty is hard to get rid off

E[y] instead of y as regressor

- Suppose:

$$\begin{aligned} y_{t+1} &= \exp\left(a_o + a_1 \ln k_t + a_2 \ln z_t\right) + u_{t+1} \\ u_{t+1} &= \text{prediction error} \end{aligned}$$

- Then you **cannot** estimate coefficients using LS based on

$$\ln\left(y_{t+1}\right) = a_o + a_1 \ln k_t + a_2 \ln z_t + u^*_{t+1}$$

- You have to use non-linear least squares

E[y] instead of y as regressor

- Suppose:

$$\begin{aligned} \mathsf{E}\left[y_{t+1}\right] &= \exp\left(a_o + a_1 \ln k_t + a_2 \ln z_t\right) + \bar{u}_{t+1} \\ \bar{u}_{t+1} &= \text{numerical error} \end{aligned}$$

- Then you **can** estimate coefficients using LS based on

$$\mathsf{E}\left[\ln\left(y_{t+1}\right)\right] = a_o + a_1 \ln k_t + a_2 \ln z_t + \bar{u}^*_{t+1}$$

- Big practical advantage

Simple ways to improve regression

1. Hermite polynomials and scaling
2. LS-Singular Value Decomposition
3. Principle components

Simple ways to improve regression

- The main underlying problem is that $X'X$ is ill conditioned which makes it difficult to calculate $X'X$
- This problem is reduced by

1. Scaling so that each variable has zero mean and unit variance
2. Hermite polynomials

Hermite polynomials; Definition

$$P_n(x) = \sum_{j=0}^{n} a_j H_j(x)$$

where the basis functions, $H_j(x)$, satisfy

$$\begin{aligned} \mathsf{E}\left[H_i(x)H_j\left(x\right)\right] &= 0 \text{ for } i \neq j \\ \text{if } x &\sim N(0,1) \end{aligned}$$

Hermite polynomials; Construction

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= x \\ H_{m+1}(x) &= xH_m(x) - mH_{m-1}(x) \text{ for } j > 1 \end{aligned}$$

This gives

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= x \\ H_2(x) &= x^2 - 1 \\ H_3(x) &= x^3 - 3x \\ H_4(x) &= x^4 - 6x^2 + 3 \\ H_5(x) &= x^5 - 10x_3 + 15x \end{aligned}$$

One tricky aspect about scaling

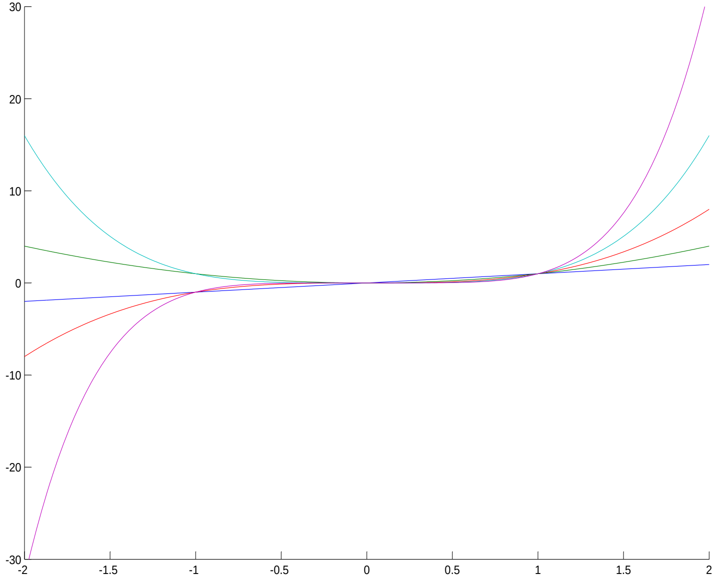
Suppose one of the explanatory variables is

$$
\begin{aligned}
x_t &= \frac{k_t - M_T}{S_T} \\
M_T &= \sum_{t=1}^{T} k_t / T \ \&\ S_T = \left(\sum_{t=1}^{T} \left(k_t - M(k_t\right)^2 / T \right)^{1/2}
\end{aligned}
$$

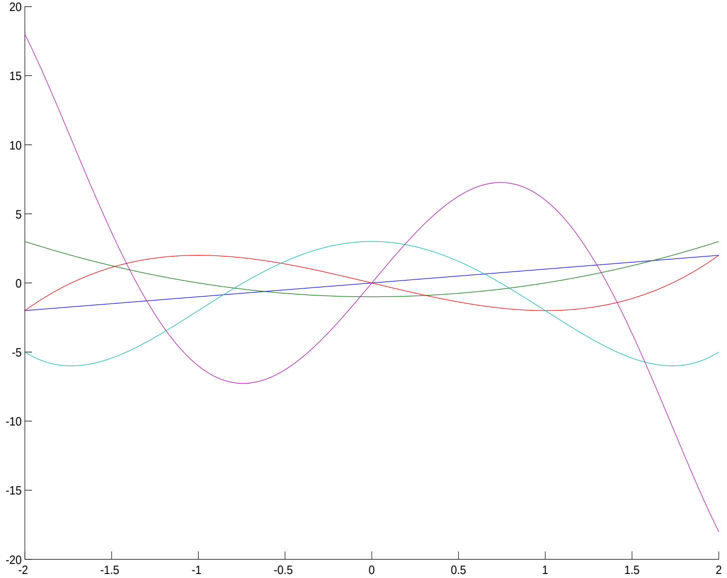
One tricky aspect about scaling

- $\Longrightarrow$ each iteration the explanatory variables change (since M and S change)
- $\Longrightarrow$ taking a weighted average of old and new coefficient is odd
- I found that convergence properties can be quite bad
  actually better without taking a weighted average,
  but that only works for well behaved models
- In principle you can avoid problem by rewriting polynomial,
  but that is tedious for higher-order
- So better to keep M_T and S_T fixed across iterations

Two graphs say it all; regular polynomials



Two graphs say it all; Hermite polynomials



LS-Singular Values Decomposition

- Goal: avoid calculating $X'X$ explicitly
- SVD of the $(T \times n)$ matrix X :

$$\begin{array}{rcl} X & = & USV' \\ U & : & (T \times n) \text{ orthogonal matrix} \\ S & : & (n \times n) \text{ diagonal matrix with singular values } s_1 \geq s_2 \geq \cdot\cdot \\ V & : & (n \times n) \text{ orthogonal matrix} \end{array}$$

- s_i is the sqrt of i^{th} eigen value

LS-Singular Values Decomposition

$$\widehat{\beta} = \left(X'X\right)^{-1} X'Y = VS^{-1}U'Y$$

- Goal: avoid calculating $X'X$ explicitly
- SVD of the $(T \times n)$ matrix X :

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- s_i is the sqrt of i^{th} eigen value

LS-Singular Values Decomposition

In Matlab

```
[U,S,V]=svd(X,0);
```

Principle components

- With many explanatory variables use principle components
  - SVD: $X = USV'$ where X is demeaned
  - Principle components: $Z = XV$
  - Properties Z_i : mean zero and variance s_i^2
- Idea: exclude principle components corresponding to lower eigenvalues
- But check with how much R^2 drops

PEA and learning

- Traditional algorithm:
  - simulate an economy using belief η_n^i
  - formulate new belief η_n^{i+1}
  - simulate *same* economy using belief η_n^{i+1}

PEA and learning

- Alternative algorithm to find *fixed point*
  - simulate T observations using belief η_n^{T-1}
  - formulate new belief η_n^T
  - generate 1 more observation
  - use $T+1$ observations to formulate new belief η^{T+1}
  - continue
- Convergence properties can be problematic

PEA and learning

- Modification of alternative algorithm is economically interesting
  - simulate T observations using belief η_n^{T-1}
  - use τ observations to formulate new belief η_n^T
  - generate 1 more observation
  - use last τ observations to formulate new belief η^{T+1}
  - continue
- Beliefs are based on limited past $\Longrightarrow$ time-varying beliefs

PEA and learning

- Suppose the model has different regimes
  - e.g. high productivity and low productivity regime
  - agents do not observe regime$\Longrightarrow$ it makes sense to use limited number of past observations
- With the above algorithm agents gradually learn new law of motion

PEA and perturbation

- True in many macroeconomic models:
    - perturbation generates accurate solution of real side of the economy
    - perturbation does not generates accurate solution of asset prices
    - real side does not at all or not much depend on asset prices
- Then solve for real economy using perturbation and for asset prices using PEA
    - one-step algorithm (no iteration needed)

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