Solving Models with Heterogeneous Expectations

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Overview

1. Current approaches to model heterogeneous expectations
2. Numerical algorithm to solve models with rational and irrational agents the right way

Modelling heterogeneous expectations

- Agent-based modelling:
  - several papers have "fundamental" or "rational" agents but terminology is very misleading
  - lack of forward looking agents makes numerical analysis very straightforward (just simulate the economy)
- Rule of thumb and rational agents
  - popular in model with New Keynesian Phillips curve
  - But NK Phillips curve has been derived under assumption of homogeneous expectations

Some papers do combine both elements

Nice examples in the literature:

- Nunes (*Macroeconomic Dynamics*, 2009)
  - rational agents and agents that learn
- Molnar (2010)
  - rational agents and backward looking agents
  - fractions of each endogenous

Nunes (2009)

NK Phillips curve:

$$\pi_t = \kappa z_t + \beta \widetilde{\mathsf{E}}_t \pi_{t+1}$$

IS curve:

$$z_t = \widetilde{\mathsf{E}}_t z_{t+1} - \sigma^{-1} \left(r_t - r_t^n - \widetilde{\mathsf{E}}_t \pi_{t+1} \right)$$

Nunes (2009)

Natural rate:

$$r_t^n = \rho r_{t-1}^n + \varepsilon_t$$

Policy:

$$r_t = \pi^* + \phi_\pi \left(\pi_t - \pi^*\right) + \phi_z z_t$$

Nunes (2009)

Nice:

- Rational agents are truly rational

Not nice:

- Nunes follows the standard approach:
    1. use a representative agent model
    2. simply replace the expectation by a weighted average of agents expectations

But how do I know these are the right relationships if agents have heterogeneous expectations

Molnar (2010)

Model:

$$\begin{aligned} p_t &= \lambda \mathsf{E}_t \left[p_{t+1} \right] + m_t \\ m_t &= \rho m_{t-1} + \varepsilon_t, \quad \rho \in [0, 1) \end{aligned}$$

Again, $\mathsf{E}_t \left[p_{t+1} \right]$ is a weighted average of the expectations of different types

New Keynesian model and aggregation

NK Phillips curve:

$$\pi_t = \beta \mathsf{E}_t \left[\pi_{t+1}\right] + \lambda y_t$$

- Question: Does this equation hold in models with heterogeneous agents with $\mathsf{E}_t \left[\pi_{t+1}\right]$ replaced by weighted average?
- Branch and McGough (2009):
  - there is a set of assumptions for which the answer is yes
  - assumptions are restrictive
  - even necessary conditions are shown to be restrictive

Alternative setup

- Model behavior of *individual* agents from the ground up
  - some rational
  - some not rational
- Explicit aggregate their behavior to get aggregate behavior
- Can we solve these models? Yes
  - using the tools learned in this course

Environment

- unit mass of firms
- half has rational expectations
- half has "type A" expectations
- for now fractions are fixed

Firm output

All firms have the same production function

$$y_i = z_i \left(z_1 n_{1,-1}^{\alpha} + z_2 n_{2,-1}^{\alpha} \right)$$

- two production processes
- n_1 and n_2 are chosen in previous period
- z_i : idiosyncratic shock
- z_1 and z_2 : aggregate (common) shocks

Exogenous random processes

$$
\begin{aligned}
z_i &= (1-\rho_i) + \rho_i z_{i,-1} + e_i, \quad e_i \sim N(0,\sigma_i^2) \\
z_1 &= (1-\rho) + \rho z_{1,-1} + e_1, \quad e_1 \sim N(0,\sigma^2) \\
z_2 &= (1-\rho) + \rho z_{2,-1} + e_2, \quad e_2 \sim N(0,\sigma^2)
\end{aligned}
$$

Problem rational agent

$$v(n_{1,-1}, n_{2,-1}, z) = \max_{n_1,n_2} \left(\begin{array}{c} z_i\left(z_1 n_{1,-1}^{\alpha} + z_2 n_{2,-1}^{\alpha}\right) - w(N_{-1}) - 0.5\eta(n - n_{-1})^2 \\ +\beta \mathsf{E}_t\left[v(n_1, n_2, z_{+1})\right] \end{array} \right)$$

where

$$\begin{array}{rcl} n & = & n_1 + n_2, \\ N & : & \text{aggregate employment} \end{array}$$

FOCs rational firm

$$\alpha\beta \mathsf{E}\left[z_{i,+1}\right]\mathsf{E}\left[z_{1,+1}\right]n_1^{\alpha-1}-\beta\mathsf{E}\left[w_{+1}\right]$$
$$-\eta\left(n-n_{-1}\right)+\eta\beta\mathsf{E}\left[n_{+1}-n\right]=0$$

$$\alpha\beta \mathsf{E}\left[z_{i,+1}\right]\mathsf{E}\left[z_{2,+1}\right]n_2^{\alpha-1}-\beta\mathsf{E}\left[w_{+1}\right]$$
$$-\eta\left(n-n_{-1}\right)+\eta\beta\mathsf{E}\left[n_{+1}-n\right]=0$$

FOCs type A firm

$$\alpha\beta\widehat{\mathsf{E}}\left[\hat{z}_{i,+1}\right]\widehat{\mathsf{E}}\left[z_{1,+1}\right]\hat{n}_1^{\alpha-1}-\beta\widehat{\mathsf{E}}\left[w_{+1}\right]$$
$$-\eta\left(\hat{n}-\hat{n}_{-1}\right)+\eta\beta\widehat{\mathsf{E}}\left[\hat{n}_{+1}-\hat{n}\right]=0$$

$$\alpha\beta\widehat{\mathsf{E}}\left[\hat{z}_{i,+1}\right]\widehat{\mathsf{E}}\left[z_{2,+1}\right]\hat{n}_2^{\alpha-1}-\beta\widehat{\mathsf{E}}\left[w_{+1}\right]$$
$$-\eta\left(\hat{n}-\hat{n}_{-1}\right)+\eta\beta\widehat{\mathsf{E}}\left[\hat{n}_{+1}-\hat{n}\right]=0$$

Labor supply

$$w = \omega_0 + \omega_1 \left(N_{-1} - 1\right)$$

$$N = \frac{\int n_i di + \int \hat{n}_i di}{2}$$

Normalization

$$\omega_0 = \alpha 0.5^{\alpha-1}$$

Steady state values:

$$\begin{aligned} n_1 &= n_2 = \hat{n}_1 = \hat{n}_2 = 0.5 \\ w &= \omega_0 \\ N &= 1 \end{aligned}$$

Solving the rational firm problem

- Rational firm's problem easy if I know ....

Solving the rational firm problem

- Rational firm's problem easy if I know ....
- dgp for wages (or N) or conditional expectations

Problem for type A

Just as easy or easier

Solving the complete problem

- Use KS iteration scheme

Solving the complete problem

- Use KS iteration scheme
- PEA

Varying fractions of types

- state-dependent switching
- probability of switching depends on profitability
- E.g.

$$
\begin{aligned}
P_{AR} &= \frac{1-\rho_p}{2} + \rho_p P_{AR,-1} + \eta \left(F_R - F_A\right) \\
P_{RA} &= \frac{1-\rho_p}{2} + \rho_p P_{RA,-1} - \eta \left(F_R - F_A\right)
\end{aligned}
$$

where

F_R : average profits made by rational firms

F_A : average profits made by type A firms

Implications for algorithms?

- What are the implications for deriving ALM?
- What are the implications for individual firm problem?

What is wrong?

```
alpha*n_1_R^(alpha-1)*z_i(+1)*z_1(+1)
=
wage_exp_R
+eta*(n_R-n_R(-1)
```

$$-(1-\mathtt{prob}) \quad *\mathtt{eta}*(\mathtt{n\_R}(+1)-\mathtt{n\_R})$$
$$-\mathtt{prob} \qquad \mathtt{eta}*(\mathtt{n\_A}(+1)-\mathtt{n\_R})$$

How to do it correctly with Dynare?

```
alpha*n_1_R^(alpha-1)*z_i(+1)*z_1(+1)
=
wage_exp_R
+eta*(n_R-n_R(-1)
```

$$
\begin{array}{ll}
-(1-\text{prob}) & *\text{eta}*(\text{n\_R}(+1)-\text{n\_R}) \\
-\text{prob} & \text{eta}*(???-\text{n\_R})
\end{array}
$$

How to do it correctly with Dynare?

Suppose `???` in the previous slide is an *endogenous* rule that I have to solve for.

Can I solve for the policy rule of `???` and `n_R` with Dynare?

How to do it correctly with Dynare?

Can I do this with *first-order* perturbation?

Switching types and valuation problems

example above:

- rational firm does think through implications of choices for his irrational self
- But there is no "valuation" problem to assess value of irrational self
- Would that be problematic?

Switching types and valuation problems

new example

- simpler environment
- different question, namely:
    - calculate firm value
    - while rationally taking into account that you could become irrational
    - fixed probability of switching

Firm problem

$$v(n_{-1},z) = \max_n \begin{array}{c} zn_{-1}^{\alpha} - wn_{-1} + 0.5\eta(n-n_{-1})^2 \\ +\beta \mathsf{E}_t \left[\begin{array}{c} (1-\rho)\, v(n,z_{+1}) \\ \rho w(n,z_{+1}) \end{array} \right] \end{array}$$

$$= \begin{array}{c} zn_{-1}^{\alpha} - wn_{-1} + 0.5\eta(n^*-n_{-1})^2 \\ +\beta \mathsf{E}_t \left[\begin{array}{c} (1-\rho)\, v(n^*,z_{+1}) \\ \rho w(n^*,z_{+1}) \end{array} \right] \end{array}$$

Switching types and valuation problems

- $v(n_{-1}, z)$: value of a rational firm according to rational agent
- $w(n_{-1}, z)$: value of a irrational firm according to a rational agent, i.e. *using rational expectations*

Rational value of irrational firm

$$w(n_{-1},z) = \begin{array}{c} zn_{-1}^{\alpha} - wn_{-1} + 0.5\eta(\hat{n} - n_{-1})^2 \\ +\beta \mathsf{E}_t \left[\begin{array}{c} (1-\rho)\, v(\hat{n}, z_{+1}) \\ \rho w(\hat{n}, z_{+1}) \end{array} \right] \end{array}$$

So again a standard problem

Idiosyncratic shocks

- Do I need them?

Learning problem

- Go back to original problem
  - but no idiosyncratic shocks
- Type A firms forecast using "least-squares" learning
- They use past observations to fit forecasting rule for
- $\widehat{\mathsf{E}}[z_1]$, $\widehat{\mathsf{E}}[z_2]$, $\widehat{\mathsf{E}}[w_{+1}]$, $\widehat{\mathsf{E}}[\hat{n}_{+1}]$

Forecasting rules

Example of forecasting rules

$$\widehat{\mathsf{E}}\left[y\right] = \hat{\omega}_0 + \hat{\omega}_1 y_{-1}$$

and $\hat{\omega}_0$ and $\hat{\omega}_1$ estimated using past observations

Weighted least-squares

$$y_t = bx_t + u_t$$

$$\begin{aligned}\hat{b}_T &= \frac{\sum_{t=1}^{T} \beta^{T-t} x_t y_t}{\sum_{t=1}^{T} \beta^{T-t} x_t^2} \\ &= \frac{\sum_{t=1}^{T} \beta^{T-t} x_t y_t}{K_T}\end{aligned}$$

Recursive weighted least-squares

- For the problem to be tractable we need recursive formulation for the forecasts made by type A firms
  - like in the Kalman filter

Recursive weighted least-squares

$$
\begin{aligned}
\hat{b}_{T+1} &= \frac{\sum_{t=1}^{T+1} \beta^{T+1-t} x_t y_t}{K_{T+1}} \\
&= \frac{\beta \sum_{t=1}^{T} \beta^{T-t} x_t y_t + x_{T+1} y_{T+1}}{K_{T+1}} \\
&= \frac{\beta K_T}{K_{T+1}} \hat{b}_T + \frac{x_{T+1} y_{T+1}}{K_{T+1}} \\
K_{T+1} &= \beta K_T + x_{T+1}^2
\end{aligned}
$$

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$$\alpha\beta\mathsf{E}\left[z_{i,+1}\right]\mathsf{E}\left[z_{1,+1}\right]n_1^{\alpha-1}-\beta\mathsf{E}\left[w_{+1}\right]$$
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$$-\eta\left(n-n_{-1}\right)+\eta\beta\mathsf{E}\left[n_{+1}-n\right]=0$$

Algorithm

- Parameterize expectations
  - which one(s)?
- Simulate economy
- Update expectation

State variables for rational agent

???

Tough problem?

- This is a standard problem
- Many state variables?
    - possibly if type A agents forecast a lot of variables
    - but maybe you don't need all as state variables to get an accurate solution

References

- Branch, W.A., and B.McGough, 2009, A New Keynesian Model with Heterogeneous Expectations, Journal of Economic Dynamics and Control.
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- Nunes, R., 2009, Learning the inflation target, Macroeconomic Dynamics.