

Learning Sunspots in Nonlinear Models

Wouter J. Den Haan
London School of Economics

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Overview

- Key question
- Particular model
- Analysis for linearized model
- Algorithm for true nonlinear model

Rational expectations equilibrium

- Let $g(x_t, \zeta_t; \eta^*, \sigma_\zeta^*)$ be a rational expectations solution, where
 - x_t is a vector with the usual state variables
 - ζ_t is the sunspot variable
 - with $E_t [\zeta_{t+1}] = 0$ and $E_t [\zeta_{t+1}^2] = (\sigma_\zeta^*)^2$
 - η^* are the function's coefficients

Beliefs

- Agents' expectations are based on the belief that

$$g(x_t, \zeta_t; \eta^*, \sigma_{\zeta}^*) = g(x_t, \zeta_t; \eta_{\text{perceived}}, \sigma_{\zeta, \text{perceived}})$$

- $\eta_{\text{perceived}}$ and $\sigma_{\zeta, \text{perceived}}$ are the coefficients of $g(\cdot)$
- agents are assumed to use the correct functional form !!!
 - framework modified below to let agents *approximate* $g(\cdot)$

Behavior with non-REE beliefs

- **Model** is such that if expectations are based on

$$g(x_t, \zeta_t; \eta_{\text{perceived}}, \sigma_{\zeta, \text{perceived}}),$$

then actual behavior is given by

$$g(x_t, \zeta_t; \eta_{\text{actual}}, \sigma_{\zeta, \text{actual}})$$

- **T-mapping**: This can be represented as

$$\begin{bmatrix} \eta_{\text{actual}} \\ \sigma_{\zeta, \text{actual}} \end{bmatrix} = T \left(\begin{bmatrix} \eta_{\text{perceived}} \\ \sigma_{\zeta, \text{perceived}} \end{bmatrix} \right)$$

Updating beliefs

- **Adaptive expectations:** Beliefs are updated iteratively using

$$\begin{bmatrix} \eta_{\text{perceived}} \\ \sigma_{\zeta, \text{perceived}} \end{bmatrix} = \begin{bmatrix} \eta_{\text{actual}} \\ \sigma_{\zeta, \text{actual}} \end{bmatrix}$$

or possibly

$$\begin{bmatrix} \eta_{\text{perceived}} \\ \sigma_{\zeta, \text{perceived}} \end{bmatrix} = (1 - \omega) \begin{bmatrix} \eta_{\text{actual}} \\ \sigma_{\zeta, \text{actual}} \end{bmatrix} + \omega \begin{bmatrix} \eta_{\text{perceived}} \\ \sigma_{\zeta, \text{perceived}} \end{bmatrix}$$

Complete iterative system

iteration i is indicated with a superscript

$$\begin{bmatrix} \eta_{\text{actual}}^i \\ \sigma_{\zeta, \text{actual}}^i \end{bmatrix} = T \left(\begin{bmatrix} \eta_{\text{perceived}}^i \\ \sigma_{\zeta, \text{perceived}}^i \end{bmatrix} \right)$$

$$\begin{bmatrix} \eta_{\text{perceived}}^{i+1} \\ \sigma_{\zeta, \text{perceived}}^{i+1} \end{bmatrix} = \begin{bmatrix} \eta_{\text{actual}}^i \\ \sigma_{\zeta, \text{actual}}^i \end{bmatrix}$$

Possible key question

- Let η^* and σ_{ζ}^* be coefficients of rational expectations solution
- possible key question:

$$\lim_{i \rightarrow \infty} \begin{bmatrix} \eta_{\text{perceived}}^i \\ \sigma_{\zeta, \text{perceived}}^i \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} \eta^* \\ \sigma^* \end{bmatrix}$$

for

$$\begin{bmatrix} \eta_{\text{perceived}}^1 \\ \sigma_{\zeta, \text{perceived}}^1 \end{bmatrix} \in I_{\eta^*, \sigma^*},$$

where I_{η^*, σ^*} is a neighborhood around $(\eta^*, \sigma_{\zeta}^*)$.

Agents cannot learn sunspot itself

- The sunspot variable, ζ_t , is chosen
 - Reason: agents cannot learn from system which variable from outside system can be added to system

Can you learn importance of sun spot?

- Linear framework with adaptive learning:
impact sunspot, F , and its standard deviation, σ_ζ , not separately identifiable
- product $F\sigma_\zeta$ also cannot be learned in standard linear model
 - If $F^1\sigma_{\zeta,\text{perceived}}^1 = 0$, then agents will never converge to a $F^*\sigma_\zeta^* > 0$
 - If $F^1\sigma_{\zeta,\text{perceived}}^1$ is small, then agents will never converge to a large σ_ζ^*

Key question

- Key question:

$$\lim_{i \rightarrow \infty} \begin{bmatrix} \eta_{\text{perceived}}^{i+1} \\ \sigma_{\zeta, \text{perceived}}^{i+1} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} \eta^* \\ \sigma^* \end{bmatrix}$$

for

$$\begin{bmatrix} \eta_{\text{perceived}}^1 \end{bmatrix} \in I_{\eta^*}, \text{ and } \sigma_{\zeta, \text{perceived}}^1 = \sigma_{\eta}^*$$

Important to distinguish

- ① Stability of $g(\cdot)$
- ② Stability of $T(\cdot)$

These are two different things

Stability of $g(\cdot)$

- stability of $g(\cdot)$ is about stability of time series

$$E \left[\lim_{t \rightarrow \infty} K_t \right] \stackrel{?}{\neq} \infty$$

- $g(\cdot)$ is stable since it is an REE
- $g(\cdot)$ is "more stable" for sunspots
 - sunspots are made possible by extra eigen values with modulus less than 1

Stability of $T(\cdot)$

- stability of $T(\cdot)$ is about stability of policy function itself
- $T(\cdot)$ tends to be complex and not so intuitive
 - $T(\cdot)$ is typically nonlinear even if $g(\cdot)$ is linear
- $T(\cdot)$ is "less stable" for sunspots in RBC type models
 - this is the stability puzzle

Particular model

- McGough, Meng, & Xue (2011) or MMX:
 - simple RBC model with externality
 - for some parameter values the model has learnable sunspots
 - key is to use a *negative* capital externality

Firm's production function

$$Y_t = A_t K_t^a H_t^b$$
$$A_t = \Lambda_A \bar{K}_t^{\alpha-a} \bar{H}_t^{\beta-b}$$

where:

- K_t and H_t are firm level variables
- \bar{K}_t and \bar{H}_t are aggregate variables (taken as given by firm)
- negative capital externality: $\alpha < a$

Firm's first-order conditions

$$\begin{aligned}R_t &= aA_tK_t^{a-1}H_t^b \\W_t &= bA_tK_t^aH_t^{b-1}\end{aligned}$$

In equilibrium: $K_t = \bar{K}_t$ and $H_t = \bar{H}_t$. Thus

$$\begin{aligned}R_t &= aA_tK_t^{a-1}H_t^b = a\Lambda_AK_t^{\alpha-1}H_t^\beta \\W_t &= bA_tK_t^aH_t^{b-1} = b\Lambda_AK_t^\alpha H_t^{\beta-1}\end{aligned}$$

Household's first-order conditions

$$C_t^{-\nu} = E_t [\rho (1 - \delta + R_{t+1}) C_{t+1}^{-\nu}]$$

$$\Lambda_H H_t^\chi = W_t C_t^{-\nu}$$

Complete model

$$C_t^{-\nu} = E_t \left[\rho \left(1 - \delta + a \Lambda_A K_{t+1}^{\alpha-1} H_{t+1} \right) C_{t+1}^{-\nu} \right]$$

$$\Lambda_H H_t^\chi = b \Lambda_A K_t^\alpha H_t^{\beta-1} C_t^{-\nu}$$

$$Y_t = \Lambda_A K_t^\alpha H_t^\beta$$

$$Y_t = C_t + K_{t+1} - (1 - \delta)K_t$$

Analytical solution steady state

- 1 $H_{SS} = K_{SS} = 1$
- 2 Choose Λ_A & Λ_H so that this is true
 - Λ_A & Λ_H do not affect the dynamics (only scale)
- 3 Solve for C_{SS} from budget constraint

Log-linearized system: Indeterminacy & sunspots

- with H_t substituted out, linearized system can be represented as follows:

$$\text{budget constraint: } \tilde{k}_{t+1} = d_k \tilde{k}_t + d_c \tilde{c}_t$$

$$\text{Euler equation: } \tilde{c}_t = b_k \tilde{k}_{t+1} + b_c \mathbf{E}_t [\tilde{c}_{t+1}]$$

- Let $\lambda_{J,1}$ and $\lambda_{J,2}$ be the two Eigenvalues of J

Linearized system: Indeterminacy & sunspots

linearized solution can be represented as follows:

$$\begin{aligned} \begin{pmatrix} \tilde{k}_{t+1} \\ \tilde{c}_{t+1} \end{pmatrix} &= J \begin{pmatrix} \tilde{k}_t \\ \tilde{c}_t \end{pmatrix} + \begin{pmatrix} 0 \\ F\zeta_{t+1} \end{pmatrix} \\ &= \begin{pmatrix} d_k & d_c \\ -\frac{b_k d_k}{b_c} & \frac{1-b_k d_c}{b_c} \end{pmatrix} \begin{pmatrix} \tilde{k}_t \\ \tilde{c}_t \end{pmatrix} + \begin{pmatrix} 0 \\ F\zeta_{t+1} \end{pmatrix} \end{aligned}$$

where $E_t[\zeta_{t+1}] = 0$ and $E_t[\zeta_{t+1}^2] = 1$

Linearized system: Indeterminacy & sunspots

- Let $\lambda_{J,1}$ and $\lambda_{J,2}$ be the two eigen values of J
- Using Jordan decomposition of J

$$\begin{pmatrix} \tilde{k}_{t+1} \\ \tilde{c}_{t+1} \end{pmatrix} = P \begin{bmatrix} \lambda_{J,1} & 0 \\ 0 & \lambda_{J,2} \end{bmatrix} P^{-1} \begin{pmatrix} \tilde{k}_t \\ \tilde{c}_t \end{pmatrix} + \begin{pmatrix} 0 \\ F\zeta_{t+1} \end{pmatrix}$$

Understanding indeterminacy

- suppose $\zeta_t = 0 \forall t$ (ignore sunspot for simplicity)
- given \tilde{k}_1 , value of \tilde{c}_1 can still be arbitrarily chosen
- \tilde{k}_2 follows from budget constraint
- \tilde{c}_2 follows from Euler equation
- that is, we are simply solving forward
- If $|\lambda_{J,1}| < 1$ and $|\lambda_{J,2}| < 1$, then this will converge
 - easy to find parameters to satisfy this condition

Understanding indeterminacy

- If $|\lambda_{J,1}| > 1$ and $|\lambda_{J,2}| < 1$, then it must be true that

$$P^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}' \begin{pmatrix} \tilde{k}_1 \\ \tilde{c}_1 \end{pmatrix} = 0$$

to ensure that series don't explode

- this pins down \tilde{c}_1 as a function of \tilde{k}_1

Forming expectations

- There are many ways in which you can formulate expectations
- We follow MMX:
 - agents use \tilde{k}_{t-1} , \tilde{c}_{t-1} , & ζ_t to make forecasts
 - \tilde{c}_t is solved from Euler equation using
 - $\hat{E}_t[\tilde{c}_t]$ instead of \tilde{c}_t to determine RHS

MMX system

budget constraint $\tilde{k}_{t+1} = d_k \tilde{k}_t + d_c \hat{E}_t [\tilde{c}_t]$

Euler equation $\tilde{c}_t = b_k \tilde{k}_{t+1} + b_c \hat{E}_t [\tilde{c}_{t+1}]$

Perceived law of motion and expectations

Perceived law of motion:

$$\tilde{c}_t = A + B\tilde{k}_{t-1} + D\tilde{c}_{t-1} + F\zeta_t$$

Expectations:

$$\begin{aligned}\hat{E}_t [\tilde{c}_t] &= A + B\tilde{k}_{t-1} + D\tilde{c}_{t-1} + F\zeta_t \\ \hat{E}_t [\tilde{c}_{t+1}] &= A + B\tilde{k}_t + DE_t [\tilde{c}_t] + F\hat{E}_t [\zeta_{t+1}] \\ &= A + B\tilde{k}_t + DE_t [\tilde{c}_t]\end{aligned}$$

Perceived & actual law of motion

if

$$\tilde{k}_t = d_k \tilde{k}_{t-1} + d_c \tilde{c}_{t-1}$$

$$\tilde{k}_{t+1} = d_k \tilde{k}_t + d_c \hat{E}_t [\tilde{c}_t]$$

$$\hat{E}_t [\tilde{c}_t] = A + B \tilde{k}_{t-1} + D \tilde{c}_{t-1} + F \zeta_t$$

$$\hat{E}_t [\tilde{c}_{t+1}] = A + B \tilde{k}_t + D \hat{E}_t [\tilde{c}_t]$$

$$\tilde{c}_t = b_k \tilde{k}_{t+1} + b_c \hat{E}_t [\tilde{c}_{t+1}]$$

then

Actual law of motion

$$\tilde{c}_t = \begin{bmatrix} (b_c(1+D) + b_k d_c) A \\ b_k (d_k^2 + d_c B + b_c B (d_k + D)) \\ b_k d_c (d_k + D) + b_c (B d_c + D^2) \\ (b_k d_c + b_c D) F \end{bmatrix}' \times \begin{bmatrix} 1 \\ \tilde{k}_{t-1} \\ \tilde{c}_{t-1} \\ \tilde{\zeta}_t \end{bmatrix}$$

T-Mapping

$$T \begin{pmatrix} A \\ B \\ D \\ F \end{pmatrix} = \begin{pmatrix} (b_c(1+D) + b_k d_c) A \\ b_k(d_k^2 + d_c B) + b_c B(d_k + D) \\ b_k d_c(d_k + D) + b_c(Bd_c + D^2) \\ (b_k d_c + b_c D) F \end{pmatrix}$$

Rational Expectations Equilibrium

$$A = 0$$

$$B = J_{21} = -\frac{b_k d_k}{b_c}$$

$$D = J_{22} = \frac{1 - b_k d_c}{b_c}$$

$$F = \text{anything}$$

Check

$$T(\text{REE}) \stackrel{?}{=} \text{REE}$$

$$T \begin{pmatrix} 0 \\ -\frac{b_k d_k}{b_c} \\ \frac{1 - b_k d_c}{b_c} \\ F \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 0 \\ -\frac{b_k d_k}{b_c} \\ \frac{1 - b_k d_c}{b_c} \\ F \end{pmatrix}$$

- This is true
 - this doesn't say much except; just a check on calculations

T-mapping and sunspot

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$$\left. \frac{\partial T}{\partial F} \right|_{\text{REE}} = 1$$

\implies the exact unit-root behavior implies that initial beliefs are simply confirmed

- *Exact* unit-root behavior is valid when
 - you are at the fixed point
 - no stochastics (use population moments)
 - exact linear model
- \implies in practice you should simply fix F and not iterate on it
 - \implies learning is about the non-sunspot coefficients

PEA and learning the sunspot

Overview of remaining material

- 1 Setting up PEA to generate first-order approximation
- 2 Use PEA to generate higher-order approximation

Model

$$C_t^{-\nu} = E_t \left[\rho \left(1 - \delta + a \Lambda_A K_{t+1}^{\alpha-1} H_{t+1}^{\beta} \right) C_{t+1}^{-\nu} \right]$$

$$\Lambda_H H_t^{\chi} = b \Lambda_A K_t^{\alpha} H_t^{\beta-1} C_t^{-\nu}$$

$$K_{t+1} = \Lambda_A K_t^{\alpha} H_t^{\beta} + (1 - \delta) K_t - C_t$$

PEA - first-order

$$C_t^{-\nu} = \exp \left\{ \eta_0 + \eta_k \ln (K_{t-1}/K_{ss}) + \eta_c \ln (C_{t-1}/C_{ss}) + \eta_\zeta \zeta_t \right\}$$

$$\Lambda_H H_t^\chi = b \Lambda_A K_t^\alpha H_t^{\beta-1} C_t^{-\nu}$$

$$K_{t+1} = \Lambda_A K_t^\alpha H_t^\beta + (1 - \delta) K_t - C_t$$

where

$$\exp \left\{ \eta_0 + \eta_k \ln (K_{t-1}/K_{ss}) + \eta_c \ln (C_{t-1}/C_{ss}) + \eta_\zeta \zeta_t \right\}$$

$$\approx E_t \left[\rho \left(1 - \delta + a \Lambda_A K_t^{\alpha-1} H_t^\beta \right) C_{t+1}^{-\nu} \right]$$

How to find eta coefficients

- η_ζ has to be fixed as explained above
- η_0 , η_k , and η_c can be used using regular PEA algorithm
- Note that

$$\ln C_t = -\frac{\eta_0}{\nu} - \frac{\eta_k}{\nu} \ln \left(\frac{K_{t-1}}{K_{ss}} \right) - \frac{\eta_c}{\nu} \ln \left(\frac{C_{t-1}}{C_{ss}} \right) - \frac{\eta}{\nu \zeta} \zeta_t$$

\implies solution from linearized system can be used as initial conditions

How to find eta coefficients continued

Iterative scheme:

- η_0^i , η_k^i , and η_c^i : coefficients at i^{th} iteration
- Simulate K_t , H_t , C_t , and

$$Z_{t+1} = \rho \left(1 - \delta + a\Lambda_A K_{t+1}^{\alpha-1} H_{t+1}^\beta \right) C_{t+1}^{-\nu}$$

-

$$\hat{\eta} = \arg \min_{\eta_0, \eta_k, \eta_c} \sum_{T_1}^T \left((z_{t+1}) - \exp \left\{ \begin{array}{l} \eta_0 + \eta_k \ln (K_{t-1}/K_{ss}) \\ + \eta_c \ln (C_{t-1}/C_{ss}) + \eta_\zeta \zeta_t \end{array} \right\} \right)^2$$

-

$$\eta^{i+1} = (1 - \omega) \hat{\eta} + \omega \eta^i$$

How to find eta coefficients continued

Comments:

- You are not allowed to take logs to get a linear regression equation!
- $0 \leq \omega < 1$: dampening factor
 - may be needed to get convergence

PEA - general setup

- Let $S_t = \{K_{t-1}, C_{t-1}\}$
- Approximation used:

$$\begin{aligned} E_t \left[\rho \left(1 - \delta + a\Lambda_A K_t^{\alpha-1} H_t^\beta \right) C_{t+1}^{-\nu} \right] \\ \approx \\ h_S(S_t; \eta_S) + \tilde{\eta} \sin \left(h_\zeta \left(\zeta_t, S_t; \eta_\zeta \right) \right) \end{aligned}$$

PEA - approximating function

- $h_S(S_t; \eta_S)$: a flexible functional form
- η_S : coefficients of $h_S(\cdot)$
- $h_\zeta(\zeta_t, S_t; \eta_\zeta)$: a flexible functional form
- η_ζ : coefficients of $h_\zeta(\cdot)$
- $\tilde{\eta}$: *FIXED* coefficient that determines maximum impact sunspot (since $|\sin| \leq 1$)
 - fixing $\tilde{\eta}$ corresponds to fixing F and σ_ζ above

PEA - finding eta coefficients

- Exactly as before
- Just a more complex nonlinear regression problem

Advantage of additive approximation

$$E_t \left[\rho \left(1 - \delta + a\Lambda_A K_t^{\alpha-1} H_t^\beta \right) C_{t+1}^{-\nu} \right] =$$

$$E_t \left[\begin{array}{l} \rho \left(1 - \delta + a\Lambda_A K_t^{\alpha-1} H_t^\beta \right) \times h_S(S_{t+1}; \eta_S) + \\ \rho \left(1 - \delta + a\Lambda_A K_t^{\alpha-1} H_t^\beta \right) \times \tilde{\eta} \sin \left(h_\zeta \left(\zeta_{t+1}, S_{t+1}; \eta_\zeta \right) \right) \end{array} \right]$$

\implies sunspot part is likely to have little effect on $E_t[\cdot]$

- this mimics linear case
- but in non-linear case $E_t \left[\tilde{\eta} \sin \left(h_\zeta \left(\zeta_{t+1}, S_{t+1}; \eta_\zeta \right) \right) \right]$ does not have to be zero, i.e., sun'spot can have first-order effects

A bit more on eta-tilde

- Approximation used is still:

$$\begin{aligned} & E_t \left[\rho \left(1 - \delta + a \Lambda_A K_t^{\alpha-1} H_t^\beta \right) C_{t+1}^{-\nu} \right] \\ & \approx h_S(S_t; \eta_S) + \tilde{\eta} \sin \left(h_\zeta \left(\zeta_t, S_t; \eta_\zeta \right) \right) \end{aligned}$$

- But go back to 1st-order approximation:

$$h_\zeta \left(\zeta_t, S_t; \eta_\zeta \right) = \eta_\zeta \zeta_t$$

A bit more on eta-tilde

- Question: What is

$$\lim_{i \rightarrow \infty} \eta_{\zeta}^i ?$$

- Exact unit-root type of linear non-stochastic setting not true
 - \implies unlikely that $\lim_{i \rightarrow \infty} \eta_{\zeta}^i = \eta_{\zeta}^1$
 - \implies likely that η_{ζ}^i will wander off
 - where to?

A bit more on eta-tilde - Case I

- Suppose that

$$\zeta_t = \begin{cases} -1 & \text{with probability } \frac{1}{2} \\ +1 & \text{with probability } \frac{1}{2} \end{cases}$$

- My experience (not a theorem):

$$\lim_{i \rightarrow \infty} \eta_{\zeta}^i = \pi/2 \text{ for large enough initial value}$$

$$\lim_{i \rightarrow \infty} \eta_{\zeta}^i = 0 \text{ for low enough initial value}$$

- Note that

$$\max_{\eta_{\zeta}} \left| \sin \left(\eta_{\zeta} \zeta_t \right) \right| = \pi/2$$

- Thus, impact sunspot is made as large as possible when $\eta_{\zeta}^i \rightarrow \pi/2$.

A bit more on eta-tilde - Case II

- Suppose that

$$\zeta_t \sim N(0, 1)$$

- Again it looks like convergence to different sunspot solutions is possible depending on initial conditions
- Much work remains to be done

Learnability

Other examples of learnable sunspots:

- Evans and McGough (2005): New Keynesian model with particular Taylor rule.
- Shea (2011): model with short-sighted managers.

References

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- Slides on Blanchard-Kahn conditions (& sunspots); available on line
- Slides on PEA; available on line