

Perturbation and LQ

Wouter J. Den Haan

University of Amsterdam

July 28, 2010

Neoclassical growth model - no uncertainty

$$\max_{\{c_t, k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma} - 1}{1 - \gamma}$$

s.t.

$$c_t + k_{t+1} = k_t^\alpha + (1 - \delta)k_t$$

k_1 is given

$$c_t^{-\gamma} = \beta c_{t+1}^{-\gamma} \left[\alpha k_{t+1}^{\alpha-1} + 1 - \delta \right]$$

Neoclassical growth model - no uncertainty

When we substitute out consumption using the budget constraint we get

$$\begin{aligned} (k_t^\alpha + (1 - \delta)k_t - k_{t+1})^{-\gamma} \\ = \\ \beta (k_{t+1}^\alpha + (1 - \delta)k_{t+1} - k_{t+2})^{-\gamma} [\alpha k_{t+1}^{\alpha-1} + 1 - \delta], \end{aligned}$$

Neoclassical growth model - no uncertainty

$$\max_{\{c_t, k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma} - 1}{1 - \gamma}$$

s.t.

$$c_t + k_{t+1} = k_t^\alpha + (1 - \delta)k_t$$

k_1 is given

$$c_t^{-\gamma} = \beta c_{t+1}^{-\gamma} \left[\alpha k_{t+1}^{\alpha-1} + 1 - \delta \right]$$

Neoclassical growth model - no uncertainty

When we substitute out consumption using the budget constraint we get

$$\begin{aligned} (k_t^\alpha + (1 - \delta)k_t - k_{t+1})^{-\gamma} \\ = \\ \beta (k_{t+1}^\alpha + (1 - \delta)k_{t+1} - k_{t+2})^{-\gamma} \left[\alpha k_{t+1}^{\alpha-1} + 1 - \delta \right] \end{aligned}$$

General specification I

$$f(x, x', y, y') = 0.$$

- $x : n_x \times 1$ vector of endogenous & exogenous state variables
- $y : n_y \times 1$ vector of endogenous choice variable

General specification II

Model:

$$f(x'', x', x) = 0$$

for a *known* function $f(\cdot)$.

Solution is of the form:

$$x' = h(x)$$

Thus,

$$F(x) \equiv f(h(h(x)) , h(x) , x) = 0 \quad \forall x$$

Neoclassical growth model again

$$f(k', k, c', c) =$$

$$\begin{bmatrix} -c^{-\gamma} + \beta (c')^{-\gamma} \left[\alpha (k')^{\alpha-1} + 1 - \delta \right] \\ -c - k' + k^{\alpha} + (1 - \delta)k \end{bmatrix}$$

Solution is of the form:

$$\begin{aligned} k' &= h(k) \\ c &= g(k) \end{aligned}$$

Thus,

$$F(k) \equiv$$

$$\begin{bmatrix} -g(k)^{-\gamma} + \beta g(h(k))^{-\gamma} \left[\alpha h(k)^{\alpha-1} + 1 - \delta \right] \\ -g(k) - h(k) + k^{\alpha} + (1 - \delta)k \end{bmatrix}$$

Neoclassical growth model again

$$f(k'', k', k) = \frac{(-k^\alpha - (1 - \delta)k - k')^{-\gamma}}{\beta ((k')^\alpha + (1 - \delta)k' - k'')^{-\gamma} [\alpha (k')^{\alpha-1} + 1 - \delta]},$$

for known values of α , δ , and γ

Solution is of the form: $k' = h(k)$

Thus

$$F(k) \equiv \frac{(-k^\alpha - (1 - \delta)k - h(k))^{-\gamma}}{\beta (h(k)^\alpha + (1 - \delta)h(k) - h(h(k)))^{-\gamma} [\alpha h(k)^{\alpha-1} + 1 - \delta]},$$

No uncertainty

With uncertainty

Global method

Linear-Quadratic

Key condition

$$F(k) = 0 \quad \forall x$$

Linear, Log-linear, $t(x)$ linear, etc

- All first-order solutions are linear in something
- Specification in last slide
 - \implies solution that is linear in the *level* of k

Linear, Log-linear, $t(x)$ linear, etc

- How to get a solution that is linear in $\tilde{k} = \ln(k)$?
- write the $f(\cdot)$ function as

$$f(\tilde{k}'', \tilde{k}', \tilde{k}) = \beta \left(\exp(\alpha \tilde{k}') + (1 - \delta) \exp(\tilde{k}') - \exp(\tilde{k}'') \right)^{-\gamma} \times \left[\alpha \exp((\alpha - 1)\tilde{k}') + 1 - \delta \right]$$

$$+ \left(-\exp(\alpha \tilde{k}) - (1 - \delta) \exp(\tilde{k}) - \exp(\tilde{k}') \right)^{-\gamma}$$

Linear, Log-linear, $t(x)$ linear, etc

- How we get a solution that is linear in $\hat{k} = t(k)$?
- Write the $f(\cdot)$ function as

$$\begin{aligned}
 f(\hat{k}'', \hat{k}', \hat{k}) &= \\
 &(- \left(t_{inv}(\hat{k}) \right)^\alpha - (1 - \delta) \left(t_{inv}(\hat{k}) \right) - \left(t_{inv}(\hat{k}') \right))^{-\gamma} + \\
 &\beta \left(\left(t_{inv}(\hat{k}') \right)^\alpha + (1 - \delta) \left(t_{inv}(\hat{k}') \right) - \left(t_{inv}(\hat{k}'') \right) \right)^{-\gamma} \times \\
 &\left[\alpha \left(t_{inv}(\hat{k}') \right)^{\alpha-1} + 1 - \delta \right]
 \end{aligned}$$

Numerical solution

Let

$$\bar{x} \text{ solve } f(\bar{x}, \bar{x}, \bar{x}) = 0$$

That is

$$\bar{x} = h(\bar{x})$$

Taylor expansion

$$\begin{aligned} h(x) &\approx h(\bar{x}) + (x - \bar{x})h'(\bar{x}) + \frac{(x - \bar{x})^2}{2}h''(\bar{x}) + \dots \\ &= \bar{x} + \bar{h}_1(x - \bar{x}) + \bar{h}_2 \frac{(x - \bar{x})^2}{2} + \dots \end{aligned}$$

- Goal is to find \bar{x} , \bar{h}_1 , \bar{h}_2 , etc.

Solving for first-order term

$$F(x) = 0 \quad \forall x$$

Implies

$$F'(x) = 0 \quad \forall x$$

Definitions

Let

$$\frac{\partial f(x'', x', x)}{\partial x''} \bigg|_{x''=x'=x=\bar{x}} = \bar{f}_1,$$

$$\frac{\partial f(x'', x', x)}{\partial x'} \bigg|_{x''=x'=x=\bar{x}} = \bar{f}_2,$$

$$\frac{\partial f(x'', x', x)}{\partial x} \bigg|_{x''=x'=x=\bar{x}} = \bar{f}_3.$$

Note that

$$\frac{\partial h(x)}{\partial x} \bigg|_{x=\bar{x}} = \left(\bar{h}_1 + \bar{h}_2(x - \bar{x}) + \dots \right) \bigg|_{x=\bar{x}} = \bar{h}_1$$

Key equation

$$F'(x) = 0 \quad \forall x$$

or

$$F'(x) = \frac{\partial f}{\partial x''} \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x} = 0$$

can be written as

$$F'(\bar{x}) = \bar{f}_1 \bar{h}_1^2 + \bar{f}_2 \bar{h}_1 + \bar{f}_3 = 0$$

- One equation to solve for \bar{h}_1

Key equation

$$F'(x) = 0 \quad \forall x$$

or

$$F'(x) = \frac{\partial f}{\partial x''} \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x} = 0$$

can be written as

$$F'(\bar{x}) = \bar{f}_1 \bar{h}_1^2 + \bar{f}_2 \bar{h}_1 + \bar{f}_3 = 0$$

- One equation to solve for \bar{h}_1
- Hopefully, the Blanchard-Kahn conditions are satisfied and there is only one sensible solution

Solving for second-order term

$$F'(x) = 0 \quad \forall x$$

Implies

$$F''(x) = 0 \quad \forall x$$

Definitions

Let

$$\frac{\partial^2 f(x'', x', x)}{\partial x'' \partial x} \Big|_{x''=x'=x=\bar{x}} = \bar{f}_{13}. \quad (1)$$

and note that

$$\frac{\partial^2 h(x)}{\partial x^2} \Big|_{x=\bar{x}} = \left(\bar{h}_2 + \bar{h}_3(x - \bar{x}) + \dots \right) \Big|_{x=\bar{x}} = \bar{h}_2. \quad (2)$$

Key equation

$$F''(x) = 0 \quad \forall x$$

or

$$\begin{aligned}
 & F''(x) = \\
 & + \left(\frac{\partial^2 f}{\partial x''^2} \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x'' \partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x'' \partial x} \right) \left(\frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} \right) \\
 & + \frac{\partial f}{\partial x''} \left(\frac{\partial h(x')}{\partial x'} \frac{\partial^2 h(x)}{\partial x^2} + \frac{\partial^2 h(x')}{\partial x'^2} \frac{\partial h(x)}{\partial x} \frac{\partial h(x)}{\partial x} \right) \\
 & + \left(\frac{\partial^2 f}{\partial x' x''} \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x'^2} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x' \partial x} \right) \frac{\partial h(x)}{\partial x} \\
 & \quad + \frac{\partial f}{\partial x'} \frac{\partial^2 h(x)}{\partial x^2} \\
 & + \left(\frac{\partial^2 f}{\partial x x''} \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x \partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x^2} \right)
 \end{aligned}$$

Key equation

Which can be written as

$$F''(\bar{x}) = \left(\bar{f}_{11} \bar{h}_1^2 + \bar{f}_{12} \bar{h}_1 + \bar{f}_{13} \right) \bar{h}_1^2 + \bar{f}_1 (\bar{h}_1 \bar{h}_2 + \bar{h}_2 \bar{h}_1)$$

+

$$\left(\bar{f}_{21} \bar{h}_1^2 + \bar{f}_{22} \bar{h}_1 + \bar{f}_{23} \right) \bar{h}_1 + \bar{f}_2 \bar{h}_2 + \left(\bar{f}_{31} \bar{h}_1^2 + \bar{f}_{32} \bar{h}_1 + \bar{f}_{33} \right) = 0$$

- One *linear* equation to solve for \bar{h}_2

Discussion

- Global or local?
- Borrowing constraints?
- Quadratic/cubic adjustment costs?

Neoclassical growth model with uncertainty

$$\max_{\{c_t, k_{t+1}\}_{t=1}^{\infty}} \mathbb{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma} - 1}{1 - \gamma}$$

s.t.

$$c_t + k_{t+1} = \exp(\theta_t) k_t^\alpha + (1 - \delta) k_t \quad (3)$$

$$\theta_t = \rho \theta_{t-1} + \sigma e_t, \quad (4)$$

General specification

$$\mathbb{E}f(x, x', y, y') = 0.$$

- $x : n_x \times 1$ vector of endogenous & exogenous state variables
- $y : n_y \times 1$ vector of endogenous choice variable
- Stochastic growth model: $y = c$ and $x = [k, \theta]$.

Essential insight #1

- Make uncertainty (captured by *one* parameter) explicit in system of equation

$$\mathbb{E}f(x, x', y, y', \sigma) = 0.$$

Solutions are of the form:

$$y = g(x, \sigma)$$

and

$$x' = h(x, \sigma) + \sigma \eta \varepsilon'$$

Neoclassical Growth Model

- For standard growth model we get

$$\mathbb{E}f([k, \theta], [k', \rho\theta + \sigma\varepsilon'], y, y') = 0$$

Solutions are of the form:

$$c = c(k, \theta, \sigma) \tag{5}$$

and

$$\begin{bmatrix} k' \\ \theta' \end{bmatrix} = \begin{bmatrix} k'(k, \theta, \sigma) \\ \rho\theta \end{bmatrix} + \sigma \begin{bmatrix} 0 \\ 1 \end{bmatrix} e'. \tag{6}$$

Essential insight #2

Perturb around y , x , and σ .

$$g(x, \sigma) = g(\bar{x}, 0) + g_x(\bar{x}, 0)(x - \bar{x}) + g_\sigma(\bar{x}, 0)\sigma + \dots$$

and

$$h(x, \sigma) = h(\bar{x}, 0) + h_x(\bar{x}, 0)(x - \bar{x}) + h_\sigma(\bar{x}, 0)\sigma + \dots$$

Goal

Let

$$\bar{g}_x = g_x(\bar{x}, 0), \bar{g}_\sigma = g_\sigma(\bar{x}, 0) \text{ and}$$

$$\bar{h}_x = h_x(\bar{x}, 0), \bar{h}_\sigma = h_\sigma(\bar{x}, 0).$$

Goal: to find

- $(n_y \times n_x)$ matrix \bar{g}_x , $(n_y \times 1)$ vector \bar{g}_σ , $(n_x \times n_x)$ matrix \bar{h}_x , $(n_x \times 1)$ vector \bar{h}_σ .
- The total of unknowns = $(n_x + n_y) \times (n_x + 1) = n \times (n_x + 1)$.

More on uncertainty

Results for first-order perturbation

- $\bar{g}_\sigma = \bar{h}_\sigma = 0$

Results for second-order perturbation

- $\bar{g}_{\sigma x} = \bar{h}_{\sigma x} = 0$, but $\bar{g}_{\sigma\sigma} \neq 0$ and $\bar{h}_{\sigma\sigma} \neq 0$

How to model discrete support?

Theory

- If the function is analytical \implies successive approximations converge towards the truth
- Theory says nothing about convergence patterns
- Theory doesn't say whether second-order is better than first
- For complex functions, this is what you have to worry about

Example with simple Taylor expansion

Truth:

$$\begin{aligned}f(x) = & -690.59 + 3202.4x - 5739.45x^2 \\& + 4954.2x^3 - 2053.6x^4 + 327.10x^5\end{aligned}$$

defined on $[0.7, 2]$

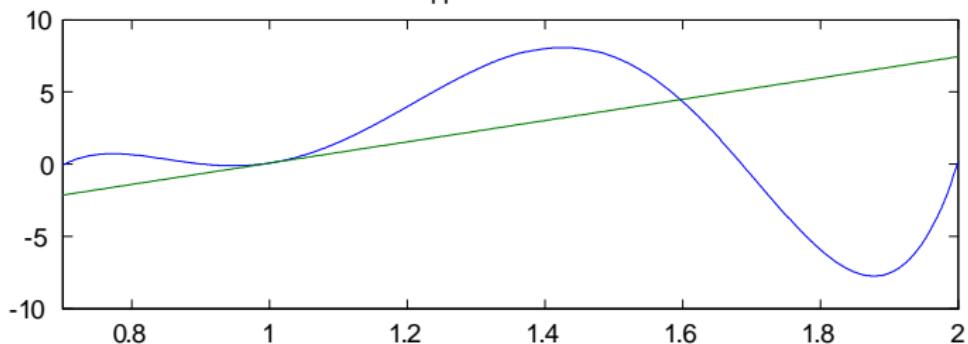
No uncertainty

With uncertainty

Global method

Linear-Quadratic

truth and level approximation of order: 1



truth and level approximation of order: 2

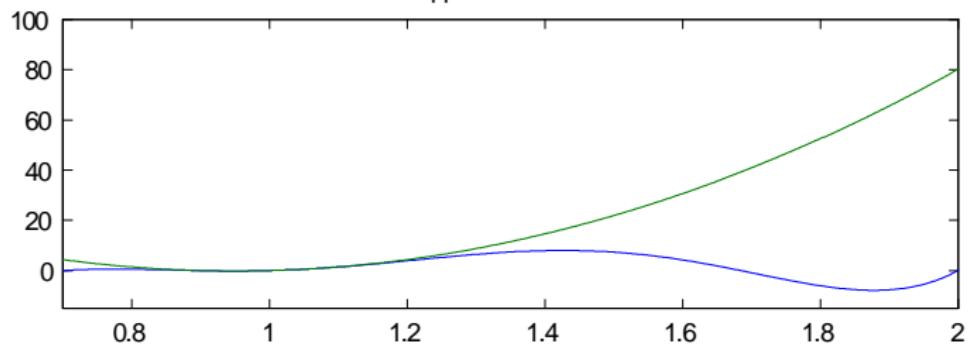


Figure: Level approximations

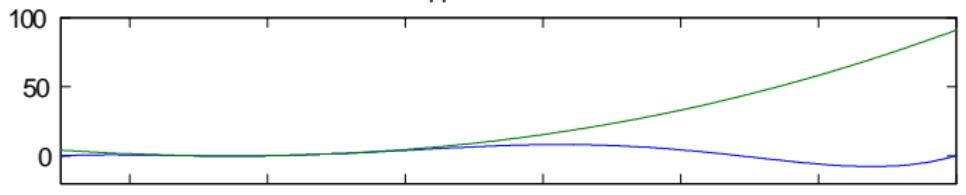
No uncertainty

With uncertainty

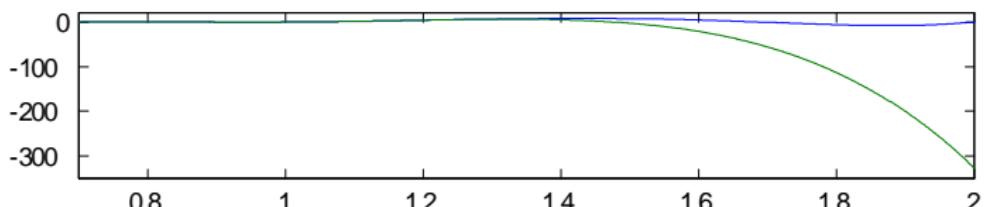
Global method

Linear-Quadratic

truth and level approximation of order: 3



truth and level approximation of order: 4



truth and level approximation of order: 5

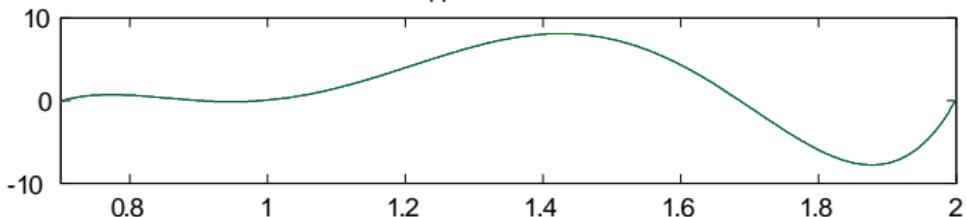


Figure: Level approximations continued

Approximation in log levels

Think of $f(x)$ as a function of $z = \log(x)$. Thus,

$$\begin{aligned}f(x) = & -690.59 + 3202.4 \exp(z) - 5739.45 \exp(2z) \\& + 4954.2 \exp(3z) - 2053.6 \exp(4z) + 327.10 \exp(5z).\end{aligned}$$

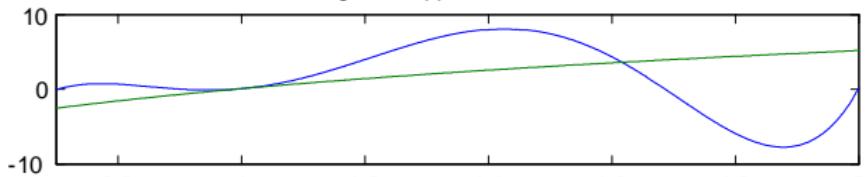
No uncertainty

With uncertainty

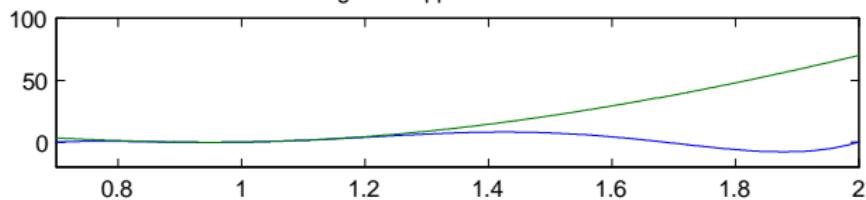
Global method

Linear-Quadratic

truth and log level approximation of order: 1



truth and log level approximation of order: 3



truth and log level approximation of order: 5

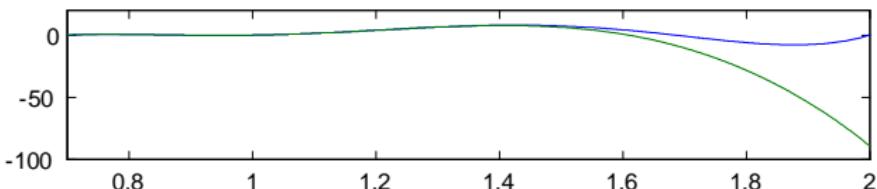


Figure: Log level approximations

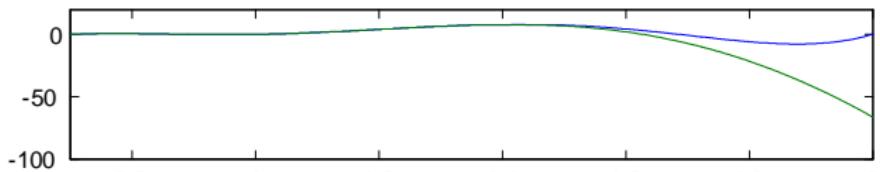
No uncertainty

With uncertainty

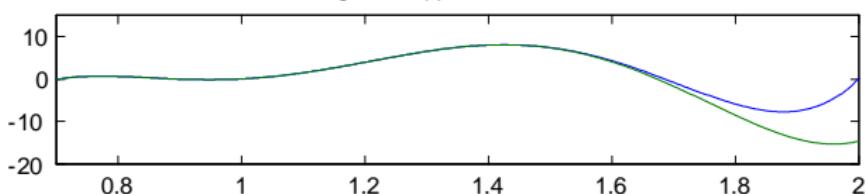
Global method

Linear-Quadratic

truth and log level approximation of order: 7



truth and log level approximation of order: 9



truth and log level approximation of order: 12

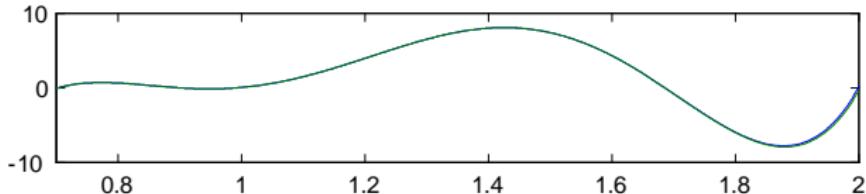


Figure: Log level approximations continued

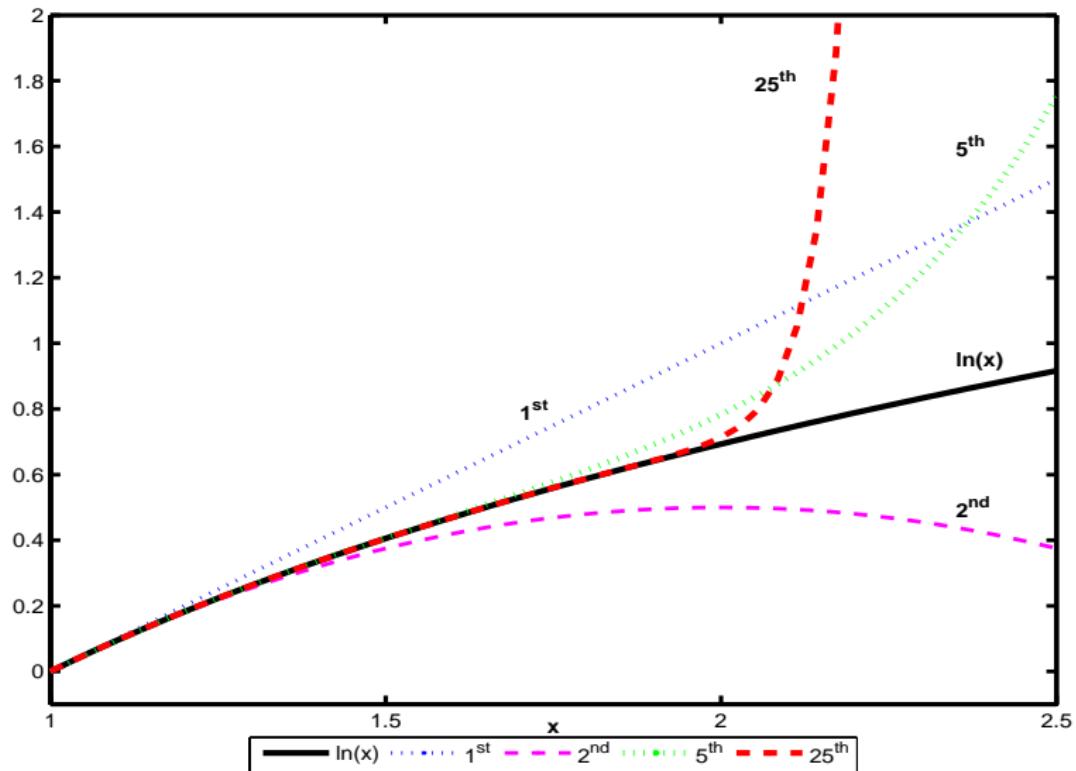
No uncertainty

With uncertainty

Global method

Linear-Quadratic

$\ln(x)$ & Taylor series expansions at $x = 1$



Are LQ & first-order perturbation the same?

True model:

$$\begin{aligned} & \max_{x,y} f(x,y,a) \\ \text{s.t. } & g(x,y,a) \leq b \end{aligned}$$

First-order conditions

$$\begin{aligned} f_x(x,y,a) + \lambda g_x(x,y,a) &= 0 \\ f_y(x,y,a) + \lambda g_y(x,y,a) &= 0 \\ g(x,y,a) &= b \end{aligned}$$

- First-order perturbation of this system will involve second-order derivatives of $g(\cdot)$
- LQ solution will not

Benigno and Woodford LQ approach

Basic Idea: Add second-order approximation to objective function
 Naive LQ approximation:

$$\begin{aligned}
 & + \bar{f}_x \tilde{x} + \bar{f}_y \tilde{y} + \bar{f}_a \tilde{a} \\
 \max_{x,y} \min_{\lambda} \quad & + \frac{1}{2} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{a} \end{bmatrix}' \begin{bmatrix} \bar{f}_{xx} & \bar{f}_{xy} & \bar{f}_{xa} \\ \bar{f}_{yx} & \bar{f}_{yy} & \bar{f}_{ya} \\ \bar{f}_{ax} & \bar{f}_{ay} & \bar{f}_{aa} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{a} \end{bmatrix} \\
 & + \lambda \begin{bmatrix} -\bar{g}_x \tilde{x} - \bar{g}_y \tilde{y} - \bar{g}_a \tilde{a} \end{bmatrix} \tag{7}
 \end{aligned}$$

Benigno and Woodford LQ approach

Step I: Take second-order approximation of constraint.

$$0 \approx +\frac{1}{2} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{a} \end{bmatrix}' \begin{bmatrix} \bar{g}_{xx} & \bar{g}_{xy} & \bar{g}_{xa} \\ \bar{g}_{yx} & \bar{g}_{yy} & \bar{g}_{ya} \\ \bar{g}_{ax} & \bar{g}_{ay} & \bar{g}_{aa} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{a} \end{bmatrix} \quad (8)$$

Benigno and Woodford LQ approach

Step 2: Multiply by steady state value of λ and add to "naive" LQ formulation:

$$\max_{x,y} \min_{\lambda} \frac{1}{2} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{a} \end{bmatrix}' \begin{bmatrix} \bar{f}_{xx} - \bar{\lambda} \bar{g}_{xx} & \bar{f}_{xy} - \bar{\lambda} \bar{g}_{xy} & \bar{f}_{xa} - \bar{\lambda} \bar{g}_{xy} \\ \bar{f}_{yx} - \bar{\lambda} \bar{g}_{yx} & \bar{f}_{yy} - \bar{\lambda} \bar{g}_{yy} & \bar{f}_{ya} - \bar{\lambda} \bar{g}_{ya} \\ \bar{f}_{ax} - \bar{\lambda} \bar{g}_{ax} & \bar{f}_{ay} - \bar{\lambda} \bar{g}_{ay} & \bar{f}_{aa} - \bar{\lambda} \bar{g}_{aa} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{a} \end{bmatrix} + \lambda \left[b - \bar{g} - \bar{g}_x \tilde{x} - \bar{g}_y \tilde{y} - \bar{g}_a \tilde{a} \right]$$