

# Perturbation and LQ

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# Neoclassical growth model - no uncertainty

$$\max_{\{c_t, k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$

s.t.

$$c_t + k_{t+1} = k_t^\alpha + (1 - \delta)k_t$$

$k_1$  is given

$$c_t^{-\gamma} = \beta c_{t+1}^{-\gamma} \left[ \alpha k_{t+1}^{\alpha-1} + 1 - \delta \right]$$

# Neoclassical growth model - no uncertainty

When we substitute out consumption using the budget constraint we get

$$\begin{aligned} & (k_t^\alpha + (1 - \delta)k_t - k_{t+1})^{-\gamma} \\ & \quad = \\ & \beta (k_{t+1}^\alpha + (1 - \delta)k_{t+1} - k_{t+2})^{-\gamma} \left[ \alpha k_{t+1}^{\alpha-1} + 1 - \delta \right], \end{aligned}$$

# Neoclassical growth model - no uncertainty

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# General specification I

$$f(x, x', y, y') = 0.$$

- $x : n_x \times 1$  vector of endogenous & exogenous state variables
- $y : n_y \times 1$  vector of endogenous choice variable

# General specification II

Model:

$$f(x'', x', x) = 0$$

for a *known* function  $f(\cdot)$ .

Solution is of the form:

$$x' = h(x)$$

Thus,

$$F(x) \equiv f(h(h(x)), h(x), x) = 0 \quad \forall x$$

# Neoclassical growth model again

$$f(k', k, c', c) =$$

$$\begin{bmatrix} -c^{-\gamma} + \beta (c')^{-\gamma} [\alpha (k')^{\alpha-1} + 1 - \delta] \\ -c - k' + k^{\alpha} + (1 - \delta)k \end{bmatrix}$$

Solution is of the form:

$$k' = h(k)$$

$$c = g(k)$$

Thus,

$$F(k) \equiv$$

$$\begin{bmatrix} -g(k)^{-\gamma} + \beta g(h(k))^{-\gamma} [\alpha h(k)^{\alpha-1} + 1 - \delta] \\ -g(k) - h(k) + k^{\alpha} + (1 - \delta)k \end{bmatrix}$$



# Neoclassical growth model again

$$f(k'', k', k) = \frac{(-k^\alpha - (1 - \delta)k - k')^{-\gamma}}{\beta ((k')^\alpha + (1 - \delta)k' - k'')^{-\gamma} [\alpha(k')^{\alpha-1} + 1 - \delta]},$$

for known values of  $\alpha$ ,  $\delta$ , and  $\gamma$

Solution is of the form:  $k' = h(k)$

Thus

$$F(k) \equiv \frac{(-k^\alpha - (1 - \delta)k - h(k))^{-\gamma}}{\beta (h(k)^\alpha + (1 - \delta)h(k) - h(h(k)))^{-\gamma} [\alpha h(k)^{\alpha-1} + 1 - \delta]},$$

# Key condition

$$F(k) = 0 \quad \forall x$$

# Linear, Log-linear, $t(x)$ linear, etc

- All first-order solutions are linear in something
- Specification in last slide
  - $\implies$  solution that is linear in the *level* of  $k$

## Linear, Log-linear, $t(x)$ linear, etc

- How to get a solution that is linear in  $\tilde{k} = \ln(k)$ ?
- write the  $f(\cdot)$  function as

$$f(\tilde{k}'', \tilde{k}', \tilde{k}) = \beta \left( \begin{aligned} & (-\exp(\alpha\tilde{k}) - (1-\delta)\exp(\tilde{k}) - \exp(\tilde{k}'))^{-\gamma} \\ & + \\ & (\exp(\alpha\tilde{k}') + (1-\delta)\exp(\tilde{k}') - \exp(\tilde{k}''))^{-\gamma} \\ & \times \\ & [\alpha \exp((\alpha-1)\tilde{k}') + 1 - \delta] \end{aligned} \right)$$

# Linear, Log-linear, $t(x)$ linear, etc

- How do we get a solution that is linear in  $\hat{k} = t(k)$ ?
- Write the  $f(\cdot)$  function as

$$\begin{aligned}
 & f(\hat{k}'', \hat{k}', \hat{k}) \\
 & = \\
 & \left( - \left( t_{inv}(\hat{k}) \right)^\alpha - (1 - \delta) \left( t_{inv}(\hat{k}) \right) - \left( t_{inv}(\hat{k}') \right) \right)^{-\gamma} + \\
 & \beta \left( \left( t_{inv}(\hat{k}') \right)^\alpha + (1 - \delta) \left( t_{inv}(\hat{k}') \right) - \left( t_{inv}(\hat{k}'') \right) \right)^{-\gamma} \times \\
 & \quad \left[ \alpha \left( t_{inv}(\hat{k}') \right)^{\alpha-1} + 1 - \delta \right]
 \end{aligned}$$

# Numerical solution

Let

$$\bar{x} \text{ solve } f(\bar{x}, \bar{x}, \bar{x}) = 0$$

That is

$$\bar{x} = h(\bar{x})$$

Taylor expansion

$$\begin{aligned} h(x) &\approx h(\bar{x}) + (x - \bar{x})h'(\bar{x}) + \frac{(x - \bar{x})^2}{2}h''(\bar{x}) + \dots \\ &= \bar{x} + \bar{h}_1(x - \bar{x}) + \bar{h}_2\frac{(x - \bar{x})^2}{2} + \dots \end{aligned}$$

- Goal is to find  $\bar{x}$ ,  $\bar{h}_1$ ,  $\bar{h}_2$ , etc.

# Solving for first-order term

$$F(x) = 0 \quad \forall x$$

Implies

$$F'(x) = 0 \quad \forall x$$

# Definitions

Let

$$\left. \frac{\partial f(x'', x', x)}{\partial x''} \right|_{x''=x'=x=\bar{x}} = \bar{f}_1,$$
$$\left. \frac{\partial f(x'', x', x)}{\partial x'} \right|_{x''=x'=x=\bar{x}} = \bar{f}_2,$$
$$\left. \frac{\partial f(x'', x', x)}{\partial x} \right|_{x''=x'=x=\bar{x}} = \bar{f}_3.$$

Note that

$$\left. \frac{\partial h(x)}{\partial x} \right|_{x=\bar{x}} = \left( \bar{h}_1 + \bar{h}_2(x - \bar{x}) + \dots \right) \Big|_{x=\bar{x}} = \bar{h}_1$$



## Key equation

$$F'(x) = 0 \quad \forall x$$

or

$$F'(x) = \frac{\partial f}{\partial x''} \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x} = 0$$

can be written as

$$F'(\bar{x}) = \bar{f}_1 \bar{h}_1^2 + \bar{f}_2 \bar{h}_1 + \bar{f}_3 = 0$$

- One equation to solve for  $\bar{h}_1$

## Key equation

$$F'(x) = 0 \quad \forall x$$

or

$$F'(x) = \frac{\partial f}{\partial x''} \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x} = 0$$

can be written as

$$F'(\bar{x}) = \bar{f}_1 \bar{h}_1^2 + \bar{f}_2 \bar{h}_1 + \bar{f}_3 = 0$$

- One equation to solve for  $\bar{h}_1$
- Hopefully, the Blanchard-Kahn conditions are satisfied and there is only one sensible solution

# Solving for second-order term

$$F'(x) = 0 \quad \forall x$$

Implies

$$F''(x) = 0 \quad \forall x$$

# Definitions

Let

$$\left. \frac{\partial^2 f(x'', x', x)}{\partial x'' \partial x} \right|_{x''=x'=x=\bar{x}} = \bar{f}_{13}. \quad (1)$$

and note that

$$\left. \frac{\partial^2 h(x)}{\partial x^2} \right|_{x=\bar{x}} = \left( \bar{h}_2 + \bar{h}_3(x - \bar{x}) + \dots \right) \Big|_{x=\bar{x}} = \bar{h}_2. \quad (2)$$

## Key equation

$$F''(x) = 0 \quad \forall x$$

or

$$\begin{aligned}
 & F''(x) = \\
 & + \left( \frac{\partial^2 f}{\partial x''^2} \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x'' \partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x'' \partial x} \right) \left( \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} \right) \\
 & \quad + \frac{\partial f}{\partial x''} \left( \frac{\partial h(x')}{\partial x'} \frac{\partial^2 h(x)}{\partial x^2} + \frac{\partial^2 h(x')}{\partial x'^2} \frac{\partial h(x)}{\partial x} \frac{\partial h(x)}{\partial x} \right) \\
 & + \left( \frac{\partial^2 f}{\partial x' x''} \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x'^2} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x' \partial x} \right) \frac{\partial h(x)}{\partial x} \\
 & \quad + \frac{\partial f}{\partial x'} \frac{\partial^2 h(x)}{\partial x^2} \\
 & + \left( \frac{\partial^2 f}{\partial x x''} \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x \partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x^2} \right)
 \end{aligned}$$

# Key equation

Which can be written as

$$\begin{aligned} F''(\bar{x}) = & \left( \bar{f}_{11} \bar{h}_1^2 + \bar{f}_{12} \bar{h}_1 + \bar{f}_{13} \right) \bar{h}_1^2 + \bar{f}_1 (\bar{h}_1 \bar{h}_2 + \bar{h}_2 \bar{h}_1) \\ & + \\ & \left( \bar{f}_{21} \bar{h}_1^2 + \bar{f}_{22} \bar{h}_1 + \bar{f}_{23} \right) \bar{h}_1 + \bar{f}_2 \bar{h}_2 + \left( \bar{f}_{31} \bar{h}_1^2 + \bar{f}_{32} \bar{h}_1 + \bar{f}_{33} \right) = 0 \end{aligned}$$

- One *linear* equation to solve for  $\bar{h}_2$

# Discussion

- Global or local?
- Borrowing constraints?
- Quadratic/cubic adjustment costs?

# Neoclassical growth model with uncertainty

$$\max_{\{c_t, k_{t+1}\}_{t=1}^{\infty}} E_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$

s.t.

$$c_t + k_{t+1} = \exp(\theta_t) k_t^\alpha + (1 - \delta) k_t \quad (3)$$

$$\theta_t = \rho \theta_{t-1} + \sigma e_t, \quad (4)$$



# General specification

$$Ef(x, x', y, y') = 0.$$

- $x : n_x \times 1$  vector of endogenous & exogenous state variables
- $y : n_y \times 1$  vector of endogenous choice variable
- Stochastic growth model:  $y = c$  and  $x = [k, \theta]$ .

# Essential insight #1

- Make uncertainty (captured by *one* parameter) explicit in system of equation

$$E f(x, x', y, y', \sigma) = 0.$$

**Solutions are of the form:**

$$y = g(x, \sigma)$$

and

$$x' = h(x, \sigma) + \sigma \eta \varepsilon'$$

# Neoclassical Growth Model

- For standard growth model we get

$$Ef([k, \theta], [k', \rho\theta + \sigma\varepsilon'], y, y') = 0$$

**Solutions are of the form:**

$$c = c(k, \theta, \sigma) \tag{5}$$

and

$$\begin{bmatrix} k' \\ \theta' \end{bmatrix} = \begin{bmatrix} k'(k, \theta, \sigma) \\ \rho\theta \end{bmatrix} + \sigma \begin{bmatrix} 0 \\ 1 \end{bmatrix} e'. \tag{6}$$

## Essential insight #2

Perturb around  $y$ ,  $x$ , and  $\sigma$ .

$$g(x, \sigma) = g(\bar{x}, 0) + g_x(\bar{x}, 0)(x - \bar{x}) + g_\sigma(\bar{x}, 0)\sigma + \dots$$

and

$$h(x, \sigma) = h(\bar{x}, 0) + h_x(\bar{x}, 0)(x - \bar{x}) + h_\sigma(\bar{x}, 0)\sigma + \dots$$

# Goal

Let

$$\bar{g}_x = g_x(\bar{x}, 0), \quad \bar{g}_\sigma = g_\sigma(\bar{x}, 0) \text{ and}$$
$$\bar{h}_x = h_x(\bar{x}, 0), \quad \bar{h}_\sigma = h_\sigma(\bar{x}, 0).$$

**Goal:** to find

- $(n_y \times n_x)$  matrix  $\bar{g}_x$ ,  $(n_y \times 1)$  vector  $\bar{g}_\sigma$ ,  $(n_x \times n_x)$  matrix  $\bar{h}_x$ ,  $(n_x \times 1)$  vector  $\bar{h}_\sigma$ .
- The total of unknowns =  
 $(n_x + n_y) \times (n_x + 1) = n \times (n_x + 1)$ .

# More on uncertainty

Results for first-order perturbation

- $\bar{g}_\sigma = \bar{h}_\sigma = 0$

Results for second-order perturbation

- $\bar{g}_{\sigma x} = \bar{h}_{\sigma x} = 0$ , but  $\bar{g}_{\sigma\sigma} \neq 0$  and  $\bar{h}_{\sigma\sigma} \neq 0$

How to model discrete support?

# Theory

- If the function is analytical  $\implies$  successive approximations converge towards the truth
- Theory says nothing about convergence patterns
- Theory doesn't say whether second-order is better than first
- For complex functions, this is what you have to worry about

# Example with simple Taylor expansion

**Truth:**

$$f(x) = -690.59 + 3202.4x - 5739.45x^2 \\ + 4954.2x^3 - 2053.6x^4 + 327.10x^5$$

defined on  $[0.7, 2]$

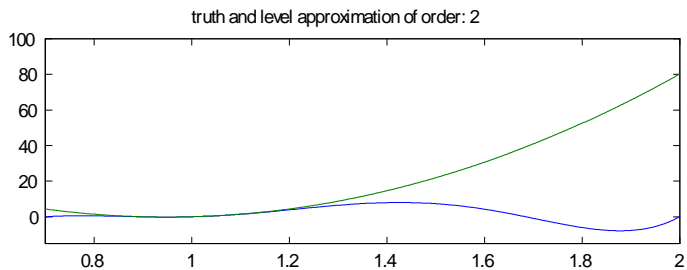
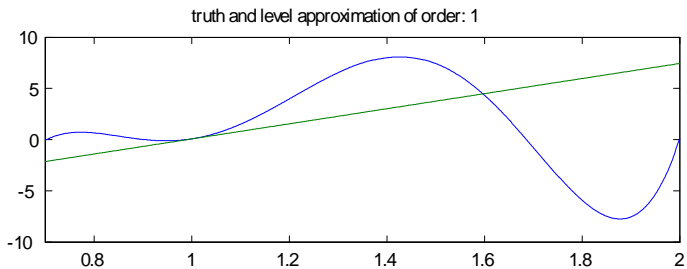


No uncertainty

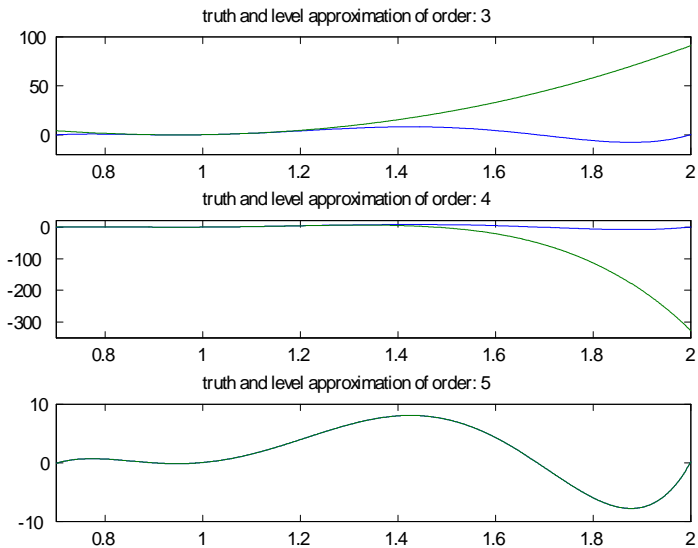
With uncertainty

Global method

Linear-Quadratic



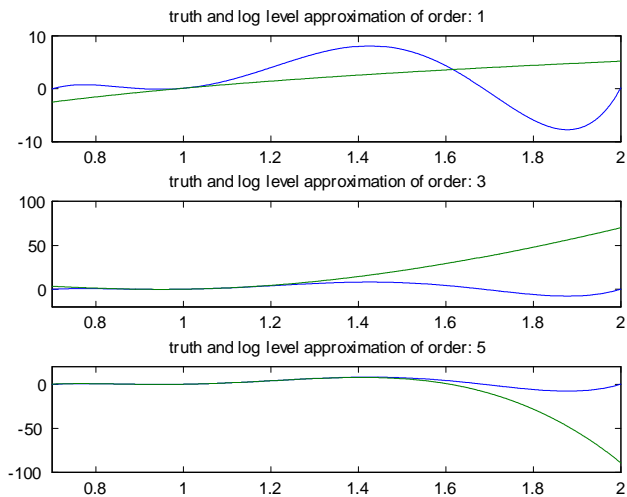
**Figure:** Level approximations

**Figure:** Level approximations continued

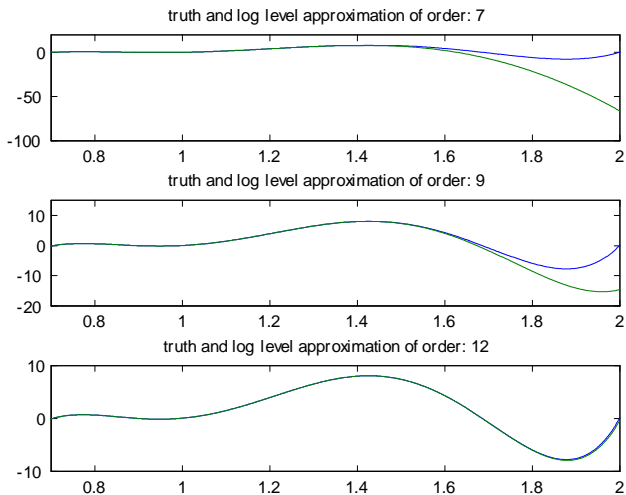
# Approximation in log levels

Think of  $f(x)$  as a function of  $z = \log(x)$ . Thus,

$$\begin{aligned} f(x) = & -690.59 + 3202.4 \exp(z) - 5739.45 \exp(2z) \\ & + 4954.2 \exp(3z) - 2053.6 \exp(4z) + 327.10 \exp(5z). \end{aligned}$$

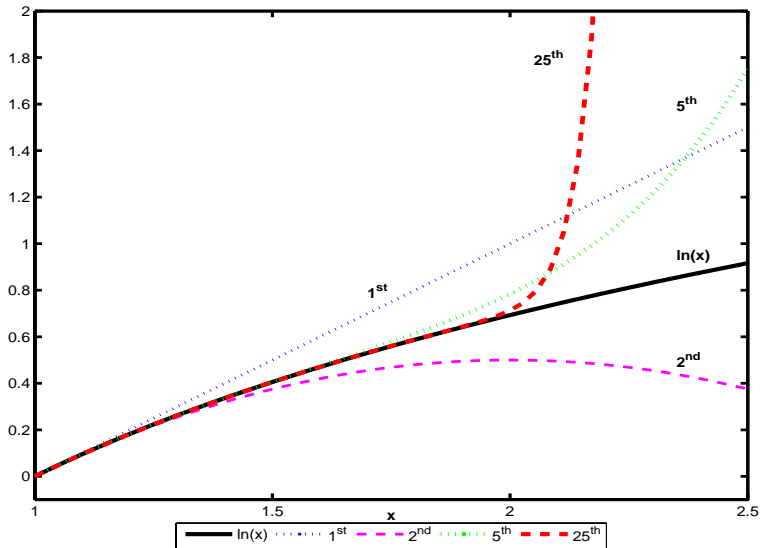


**Figure:** Log level approximations



**Figure:** Log level approximations continued

# $\ln(x)$ & Taylor series expansions at $x = 1$



# Are LQ & first-order perturbation the same?

True model:

$$\begin{aligned} & \max_{x,y} f(x, y, a) \\ \text{s.t. } & g(x, y, a) \leq b \end{aligned}$$

First-order conditions

$$\begin{aligned} f_x(x, y, a) + \lambda g_x(x, y, a) &= 0 \\ f_y(x, y, a) + \lambda g_y(x, y, a) &= 0 \\ g(x, y, a) &= b \end{aligned}$$

- First-order perturbation of this system will involve second-order derivatives of  $g(\cdot)$
- LQ solution will not

# Benigno and Woodford LQ approach

Basic Idea: Add second-order approximation to objective function

Naive LQ approximation:

$$\begin{aligned}
 & + \bar{f}_x \tilde{x} + \bar{f}_y \tilde{y} + \bar{f}_a \tilde{a} \\
 \max_{x,y} \min_{\lambda} & + \frac{1}{2} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{a} \end{bmatrix}' \begin{bmatrix} \bar{f}_{xx} & \bar{f}_{xy} & \bar{f}_{xa} \\ \bar{f}_{yx} & \bar{f}_{yy} & \bar{f}_{ya} \\ \bar{f}_{ax} & \bar{f}_{ay} & \bar{f}_{aa} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{a} \end{bmatrix} \\
 & + \lambda \begin{bmatrix} -\bar{g}_x \tilde{x} - \bar{g}_y \tilde{y} - \bar{g}_a \tilde{a} \end{bmatrix}
 \end{aligned} \tag{7}$$



# Benigno and Woodford LQ approach

Step I: Take second-order approximation of constraint.

$$0 \approx \bar{g}_x \tilde{x} + \bar{g}_y \tilde{y} + \bar{g}_a \tilde{a} + \frac{1}{2} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{a} \end{bmatrix}' \begin{bmatrix} \bar{g}_{xx} & \bar{g}_{xy} & \bar{g}_{xa} \\ \bar{g}_{yx} & \bar{g}_{yy} & \bar{g}_{ya} \\ \bar{g}_{ax} & \bar{g}_{ay} & \bar{g}_{aa} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{a} \end{bmatrix} \quad (8)$$

# Benigno and Woodford LQ approach

Step 2: Multiply by steady state value of  $\lambda$  and add to "naive" LQ formulation:

$$\max_{x,y} \min_{\lambda} \frac{1}{2} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{a} \end{bmatrix}' \begin{bmatrix} \bar{f}_{xx} - \bar{\lambda}\bar{g}_{xx} & \bar{f}_{xy} - \bar{\lambda}\bar{g}_{xy} & \bar{f}_{xa} - \bar{\lambda}\bar{g}_{xy} \\ \bar{f}_{yx} - \bar{\lambda}\bar{g}_{yx} & \bar{f}_{yy} - \bar{\lambda}\bar{g}_{yy} & \bar{f}_{ya} - \bar{\lambda}\bar{g}_{ya} \\ \bar{f}_{ax} - \bar{\lambda}\bar{g}_{ax} & \bar{f}_{ay} - \bar{\lambda}\bar{g}_{ay} & \bar{f}_{aa} - \bar{\lambda}\bar{g}_{aa} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{a} \end{bmatrix} \\ + \lambda \left[ b - \bar{g} - \bar{g}_x \tilde{x} - \bar{g}_y \tilde{y} - \bar{g}_a \tilde{a} \right]$$