Simulating models with heterogeneous agents

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Individual agent

- Subject to employment shocks ($e_{i,t} \in \{0,1\}$)
- Incomplete markets
  - only way to save is through holding capital
  - borrowing constraint $k_{i,t+1} \geq 0$
- Competitive firms, thus competitive prices
  - $w_t = (1-\alpha)\, z_t \left(\frac{K_t}{\bar{l}L_t}\right)^{\alpha}$
  - $r_t = \alpha z_t \left(\frac{K_t}{\bar{l}L_t}\right)^{\alpha-1}$

Individual agent

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} E \sum_{t=0}^{\infty} \beta^t \ln(c_{i,t})$$

s.t.

$$c_{i,t} + k_{i,t+1} = r_t k_{i,t} + (1-\tau_t) w_t \bar{l} e_{i,t} + \mu w_t (1 - e_{i,t}) + (1-\delta) k_{i,t}$$

$$k_{i,t+1} \geq 0$$

Laws of motion

- z_t can take on two values
- e_t can take on two values
- probability of being (un)employed depends on z_t
- transition probabilities are such that unemployment rate only depends on current z_t
- Thus $u_t = u^b$ if $z_t = z^b$ and $u_t = u^g$ if $z_t = z^g$ with $u^b > u^g$.

Complexity of individual problem

- for given process of r_t and w_t this is a relatively simple problem
- state variables?
- constraint?

What aggregate variables do agents care about?

- r_t and w_t
- They only depend on aggregate capital stock and z_t
- !!! This is not true in general for equilibrium prices
- Agents are interested in all information that forecasts K_t
- Thus, complete cross-sectional distribution of employment status and capital levels matters

Equilibrium

- Continuum of agents
- Individual policy functions that solve the agent's maximization problem
- A wage and a rental rate given by equations above.
- A transition law for the cross-sectional distribution of capital, that is consistent with the investment policy function.
  - $f_t =$ beginning-of-period cross-sectional distribution of capital and the employment status after the employment status has been realized.

$$f_{t+1} = \Upsilon(z_{t+1}, z_t, f_t)$$

Two different ways to go

- Simulate a panel with a large number of agents
  - This uses Monte Carlo integration to calculate cross-sectional moments
- Use tools from numerical literature
  - grid method that does not require the inverse of the policy function
  - grid method that requires the inverse of the policy function
  - non-grid method

What is given?

- A policy function $k'(k_{i,t}, e_{i,t}, s_t)$
  - s_t: the aggregate state variables
- initial distribution for $t = 1$
  - characterizes the density of capital holdings of the employed and unemployed.

Grid method I

- Fine grid with nodes: κ_i, $i = 0, 1, \cdots, \chi$
- Only mass *AT* grid points
    - $p^e_{i,t}$: mass of agents with $k^e_t = \kappa_i$, $i = 0, 1, \cdots, \chi$
    - e : employment status
    - no mass in between grid points
- If $k'_i \geq 0$ is binding $\Longrightarrow p^e_{0,t} > 0$ (and CDF has some jumps at other points)

Grid method I

- Fix employment status
  - remain within the period t for now
- Nodes correspond with *beginning-of-period* t distribution

Grid method I

- focus on node j with mass $p_t^{e,j}$ and capital value κ_j
- find i such that $k'(\kappa_j, e, \cdot)$ satisfies

$$\kappa_{i-1} < k'(\kappa_j, e, \cdot) \leq \kappa_i$$

- if $k'(\kappa_j, e, \cdot) > \kappa_\chi$, $i = \chi$

Grid method I

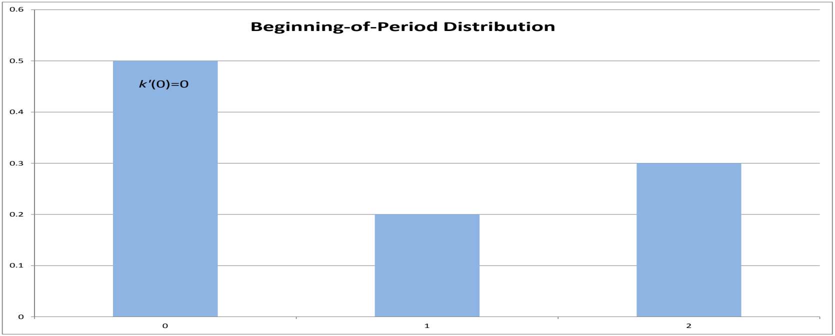
- Set end-of-period fractions:

$$f_t^{e,i} = 0 \quad \forall i$$

- Go through all nodes and allocate beginning-of-period $p_t^{e,j}$ to end-of-period $f_t^{e,i}$:

$$\text{if } k'(\kappa_j, e, \cdot) \leq \kappa_\chi \text{ then } \begin{array}{c} \omega_t^{i,j} = \frac{k'(\kappa_j, e, \cdot) - \kappa_{i-1}}{\kappa_i - \kappa_{i-1}} \\ f_t^{e,i-1} = f_t^{e,i-1} + p_t^{e,j}\left(1 - \omega_t^{i,j}\right) \\ f_t^{e,i} = f_t^{e,i} + p_t^{e,j}\omega_t^{i,j} \end{array}$$

$$\text{if } k'(\kappa_j, e, \cdot) > \kappa_\chi \text{ then } f_t^{e,\chi} = f_t^{e,\chi} + p_t^{e,j}$$


Beginning-of-Period Distribution
k'(0)=0
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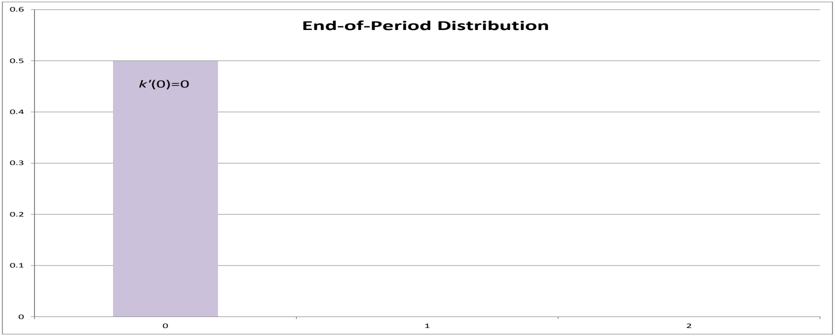
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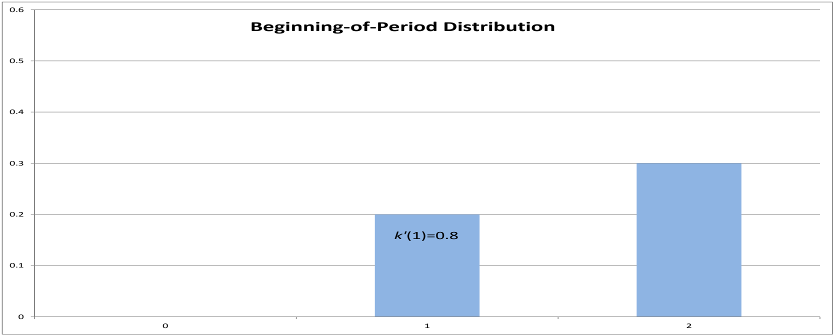
Beginning-of-Period Distribution
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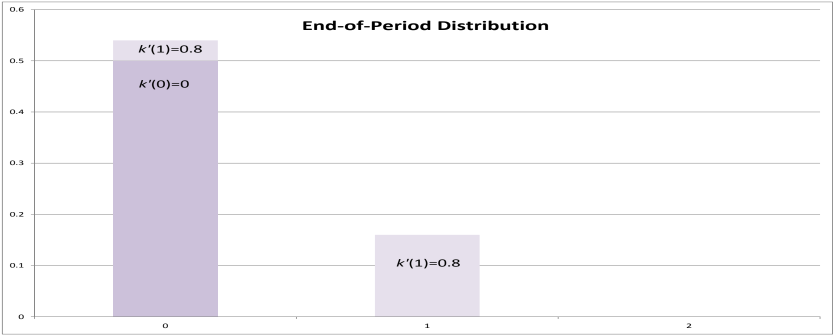
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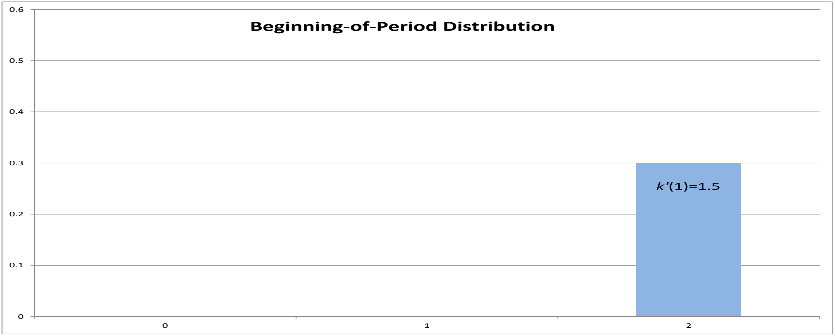
Beginning-of-Period Distribution
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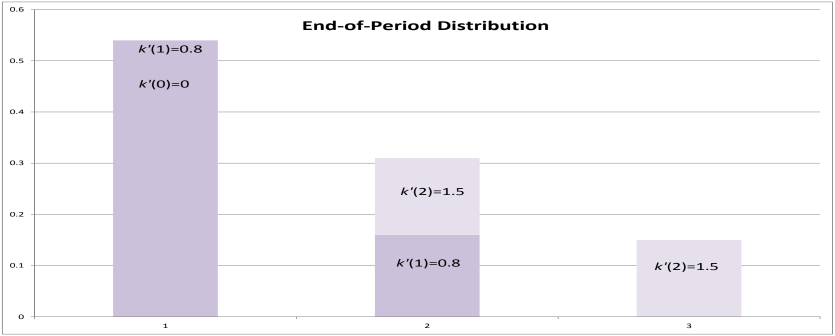
k'(1)=1.5


End-of-Period Distribution
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k'(1)=0.8
k'(0)=0
k'(2)=1.5
k'(1)=0.8
k'(2)=1.5

Grid method I

- Use transition laws to go from end-of-period t to beginning-of-period $t+1$ distribution

Next period's distribution?

- $g_{e_t e_{t+1} z_t z_{t+1}}$ = mass of agents with employment status e_t that have employment status e_{t+1}, conditional on the values of z_t and z_{t+1}
- For each combination of values of z_t and z_{t+1} we have

$$g_{00z_t z_{t+1}} + g_{01z_t z_{t+1}} + g_{10z_t z_{t+1}} + g_{11z_t z_{t+1}} = 1$$

We then have

$$p_{t+1}^{e,i} = \frac{g_{0ez_t z_{t+1}}}{g_{0ez_t z_{t+1}} + g_{1ez_t z_{t+1}}} f_t^{0,i} + \frac{g_{1ez_t z_{t+1}}}{g_{0ez_t z_{t+1}} + g_{1ez_t z_{t+1}}} f_t^{1,i}$$

Grid method II

- Distribution uniformly distributed between grid points
  - CDFs: two piece-wise linear splines, $P_t^{e=0}(k)$ and $P_t^{e=1}(k)$

Grid method II

- Calculate the end-of-period distribution as follows
    - nodes correspond to the *end-of-period* distribution
    - go through the nodes, κ_i, one by one
    - calculate the beginning-of-period capital stock at which the agent would have chosen the value at the grid point, $x_t^{e,i} = k^{\prime,inv}(\kappa_i, e_t, s_t)$
    - CDF at grid point is equal to $P_t^e(x_t^{e,i})$ (Note the two time subscripts)
    - $$F_t^{e,i} = \int_0^{x_t^{e,i}} dP_t^e(k) = \sum_{i=0}^{\overline{i_e}} p_t^{e,i} + \frac{x_t^{e,i} - \kappa_{\overline{i_e}}}{\kappa_{1+\overline{i_e}} - \kappa_{\overline{i_e}}} p_t^{e,\overline{i_e}+1},$$

      where $\overline{i_e} = \bar{i}(x_t^{e,i})$ is the largest value of i such that $\kappa_i \leq x_t^{e,i}$
- Calculate next period's beginning-of-period distribution using the transtion laws

Next period's distribution?

$$P_{t+1}^{e,i} = \frac{g_{0ez_tz_{t+1}}}{g_{0ez_tz_{t+1}} + g_{1ez_tz_{t+1}}} F_t^{0,i} + \frac{g_{1ez_tz_{t+1}}}{g_{0ez_tz_{t+1}} + g_{1ez_tz_{t+1}}} F_t^{1,i}$$

and

$$\begin{aligned} p_{t+1}^{e,0} &= P_{t+1}^{e,0} \\ p_{t+1}^{e,i} &= P_{t+1}^{e,i} - P_{t+1}^{e,i-1} \end{aligned}$$

Parameterized cross-sectional distribution

- Grid methods are likely to require a lot of nodes (1000 is typical)
- For some procedures that is costly
- Not clear you need such precise information

Parameterized cross-sectional distribution

- Grid methods are likely to require a lot of nodes (1000 is typical)
- For some procedures that is costly
- Not clear you need such precise information
- Algan, Allais, and Den Haan (2006) propose to use polynomials
    - $P(k;\rho_t)$ is a polynomial with in period $t=1$ coefficients equal to ρ_1
    - Using Simpson quadrature to calculate end-of-period moments
    - Use transition laws to calculate next period's beginning-of-period moments
- You need a way to find ρ_2 given values for moments in period 2

Fitting a distribution given moments: Approach I

- Find N elements of ρ such that

$$\int_0^\infty [k - m(1)] \; P(k;\rho)dk = 0$$

$$\int_0^\infty \left[(k - m(1))^2 - m(2)\right] \; P(k;\rho)dk = 0$$

$$\dots$$

$$\int_0^\infty \left[(k - m(1))^N - m(N)\right] \; P(k;\rho)dk = 0$$

$$\int_0^\infty P(k;\rho)dk = 1$$

Fitting a distribution given moments: Approach II

- Use an alternative functional form

- $$P(k;\rho) = \rho_0 \exp\left(\begin{array}{c} \rho_1 \left[k - m(1)\right] + \\ \rho_2 \left[(k - m(1))^2 - m(2)\right] + \cdots + \\ \rho_N \left[(k - m(1))^N - m(N)\right] \end{array} \right)$$

Fitting a distribution given moments: Approach II

- Find coefficients ρ using
-
$$\min_{\rho_1,\rho_2,\cdots,\rho_N} \int_0^\infty P(k,\rho)dk$$
- The first-order conditions correspond exactly to the condition that the first N moments of $P(k,\rho)$ should correspond to the set of specified moments.
- ρ_0 is determined by the condition that the density integrates to one.

Fitting a distribution given moments: Approach II

- The Hessian (times ρ_0) is given by

$$\int_0^\infty X\left(m(1),\cdots,m(N)\right) X\left(m(1),\cdots,m(N)\right)' P(k,\rho)dk, \tag{1}$$

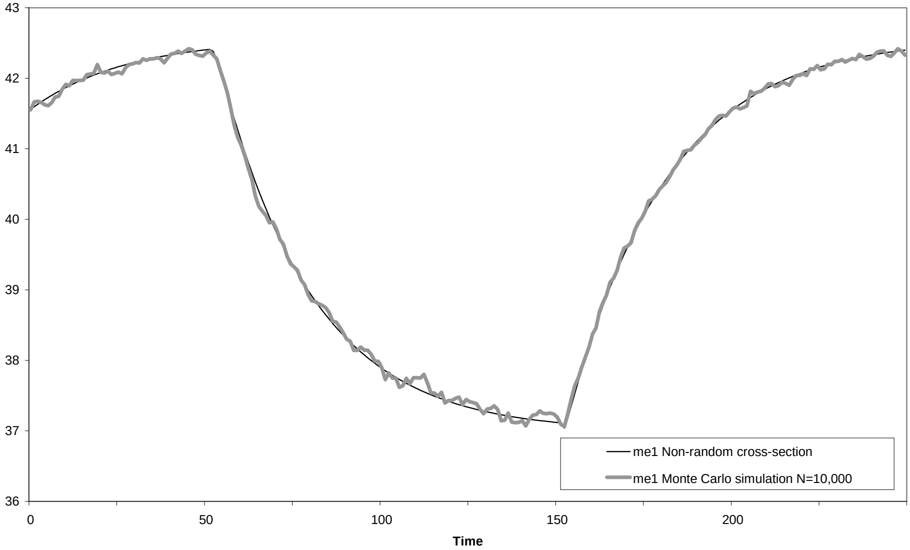
where X is an $(N \times 1)$ vector and the i^{th} element is given by

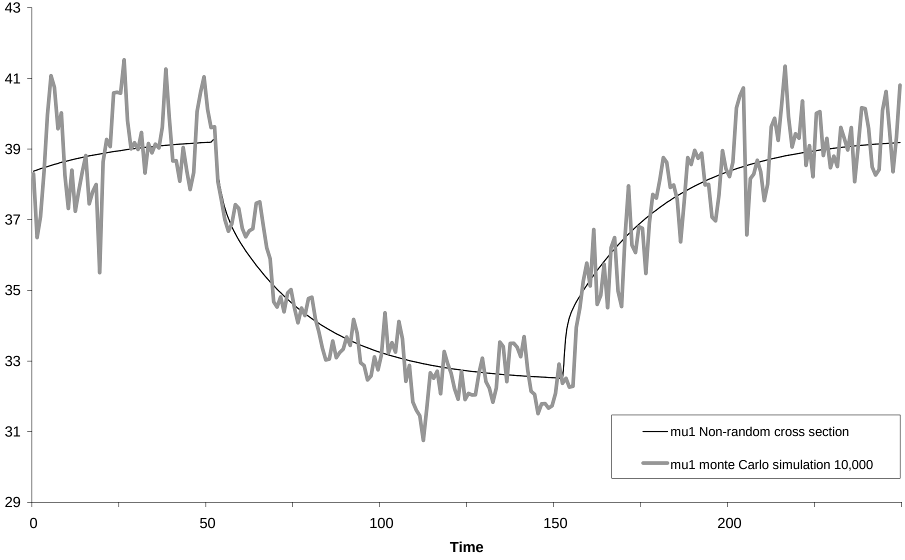
$$\begin{array}{ll} (k - m(1)) & \text{for } i = 1 \\ (k - m(1))^i - m(i) & \text{for } i > 1 \end{array} \tag{2}$$

Does it make a difference?

- Numerical procedure with a continuum of agents
- What if you really do like to simulate a panel with a finite number of agents?
  - Impose truth as much as possible: if you have 10,000 agents have exactly 400 (1,000) agents unemployed in a boom (recession)
  - Even then sampling noise is non-trivial
  - This is done in the graphs below, but still the results are not accurate

Simulation and sampling noise

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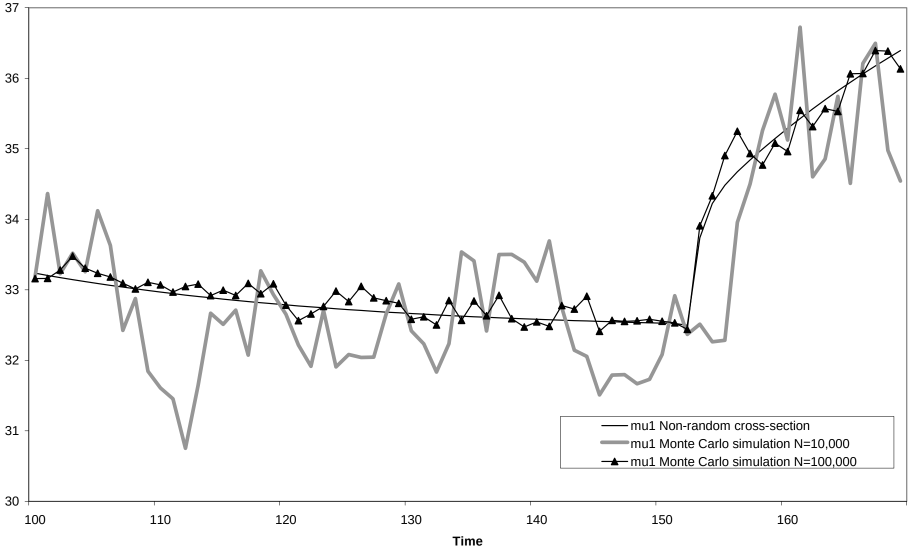
mu1 monte Carlo simulation 10,000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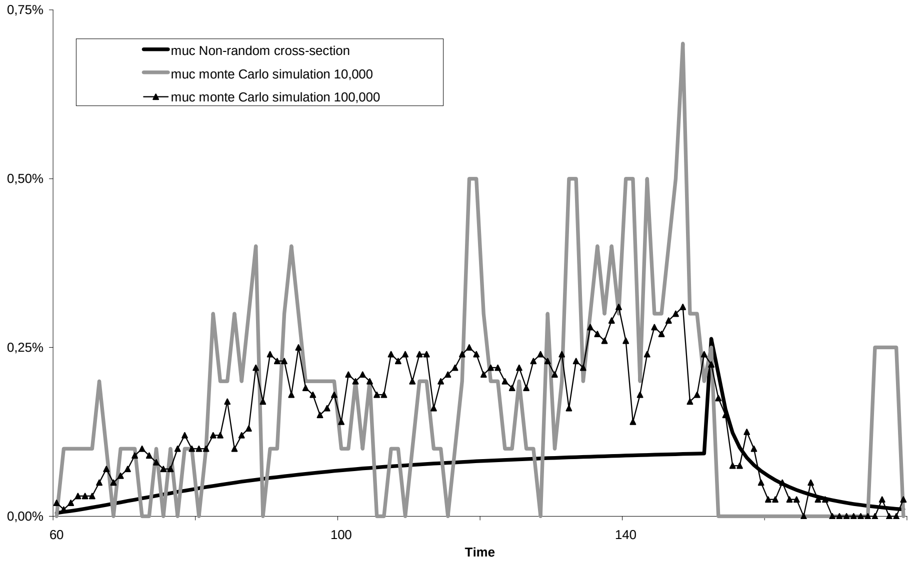
mu1 Monte Carlo simulation N=10,000
mu1 Monte Carlo simulation N=100,000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Time

References

- Algan, Y., O. Allais, W.J. Den Haan, 2008, Solving heterogeneous-agent models with parameterized cross-sectional distributions, Journal of Economic Dynamics and Control
  - simulates and solves model with parameterized expectation
- Den Haan, W. J., and P. Rendahl, 2010, Solving the incomplete markets model with aggregate uncertainty using explicit aggregation, Journal of Economic Dynamics and Control
  - article that develops Xpa
- Young, E. R., 2010, Solving the incomplete markets model with aggregate uncertainty using the Krusell-Smith algorithm and non-stochastic simulations,Journal of Economic Dynamics and Control
  - introduces the first grid method