Learning in Macroeconomic Models

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Overview

- A bit of history of economic thought
- How expectations are formed can matter *in the long run*
  - Seignorage model
- Learning without feedback
- Learning with feedback
  - Adaptive learning
  - Least-squares learning
  - Bayesian versus least-squares learning
  - Decision theoretic foundation of Adam & Marcet

Overview continued

Topics

- Learning & PEA
- Learning & sun spots

Why are expectations important?

- Most economic problems have intertemporal consequences
  - $\Longrightarrow$ future matters
- Moreover, future is uncertain
- Characteristics/behavior other agents can also be uncertain
  - $\Longrightarrow$ expectations can also matter in one-period problems

History of economic thought

- adaptive expectations:

$$\widehat{\mathsf{E}}_t\left[x_{t+1}\right] = \widehat{\mathsf{E}}_{t-1}\left[x_t\right] + \omega\left(x_t - \widehat{\mathsf{E}}_{t-1}\left[x_t\right]\right)$$

- very popular until the 70s

History of economic thought

problematic features of adaptive expectations:

- agents can be systematically wrong
- agents are completely passive:
  - $\widehat{\mathsf{E}}_t\left[x_{t+j}\right], j \geq 1$ only changes (at best) when x_t changes
  - $\Longrightarrow$ Pigou cycles are not possible
  - $\Longrightarrow$ model predictions *under*estimate speed of adjustment (e.g. for disinflation policies)

History of economic thought

problematic features of adaptive expectations:

- adaptive expectations about $x_{t+1} \neq$ adaptive expecations about Δx_{t+1}
  - (e.g. price level versus inflation)
- why wouldn't (some) agents use existing models to form expectations?
- expectations matter but still no role for randomness (of future realizations)
  - so no reason for buffer stock savings
  - no role for (model) uncertainty either

History of economic thought

rational expectations became popular because:

- agents are no longer passive machines, but forward looking
  - i.e., agents *think* through what could be consequences of their own actions and those of others (in particular government)
- consistency between model predictions and of agents being described
- randomness of future events become important
  - e.g., $\mathsf{E}_t\left[c_{t+1}^{-\gamma}\right] \neq \left(\mathsf{E}_t\left[c_{t+1}\right]\right)^{-\gamma}$

History of economic thought

problematic features of rational expectations

- agents have to know *complete* model
    - make correct predictions about all possible realizations
        - on *and* off the equilibrium path
- costs of forming expecations are ignored
- how agents get rational expectations is not explained

History of economic thought

problematic features of rational expectations

- makes analysis more complex
  - behavior this period depends on behavior tomorrow for all possible realizations
  - $\Longrightarrow$ we have to solve for policy *functions*, not just simulate the economy

Expectations matter

- Simple example to show that how expectations are formed can matter in the long run
    - See Adam, Evans, & Honkapohja (2006) for a more elaborate analysis

Model

- Overlapping generations
- Agents live for 2 periods
- Agents save by holding money
- No random shocks

Model

$$\max_{c_{1,t},c_{2,t}} \ln c_{1,t} + \ln c_{2,t}$$

s.t.

$$c_{2,t} \leq 1 + \frac{P_t}{P^e_{t+1}} \left(2 - c_{1,t}\right)$$

no randomness $\Longrightarrow$ we can work with expected value of variables instead of expected utility

Agent's behavior

First-order condition:

$$\frac{1}{c_{1,t}} = \frac{P_t}{P^e_{t+1}} \frac{1}{c_{2,t}} = \frac{1}{\pi^e_{t+1}} \frac{1}{c_{2,t}}$$

Solution for consumption:

$$c_{1,t} = 1 + \pi^e_{t+1}/2$$

Solution for real money balance (=savings):

$$m_t = 2 - c_{1,t} = 1 - \pi^e_{t+1}/2$$

Money supply

$$\overline{M}_t^s = \overline{M}$$

Equilibrium

Equilibrium in period t implies

$$
\begin{aligned}
\overline{M} &= M_t \\
\overline{M} &= P_t \left(1 - \pi^e_{t+1}/2\right) \\
P_t &= \frac{\overline{M}}{1 - \pi^e_{t+1}/2}
\end{aligned}
$$

Equilibrium

Combining with equilibrium in period $t-1$ gives

$$\pi_t = \frac{P_t}{P_{t-1}} = \frac{1-\pi_t^e/2}{1-\pi_{t+1}^e/2}$$

Thus: π_t^e & $\pi_{t+1}^e \Longrightarrow$ money demand $\Longrightarrow$ actual inflation π_t

Rational expectations solution

Optimizing behavior & equilibrium:

$$\frac{P_t}{P_{t-1}} = T(\pi_t^e, \pi_{t+1}^e)$$

Rational expectations equilibrium (REE):

$$\begin{aligned}
\pi_t &= \pi_t^e \\
&\Longrightarrow \\
\pi_t &= T(\pi_t, \pi_{t+1}) \\
&\Longrightarrow \\
\pi_{t+1} &= 3 - \frac{2}{\pi_t} \\
\pi_{t+1} &= R\left(\pi_t\right)
\end{aligned}$$

Multiple steady states

- There are two solutions to

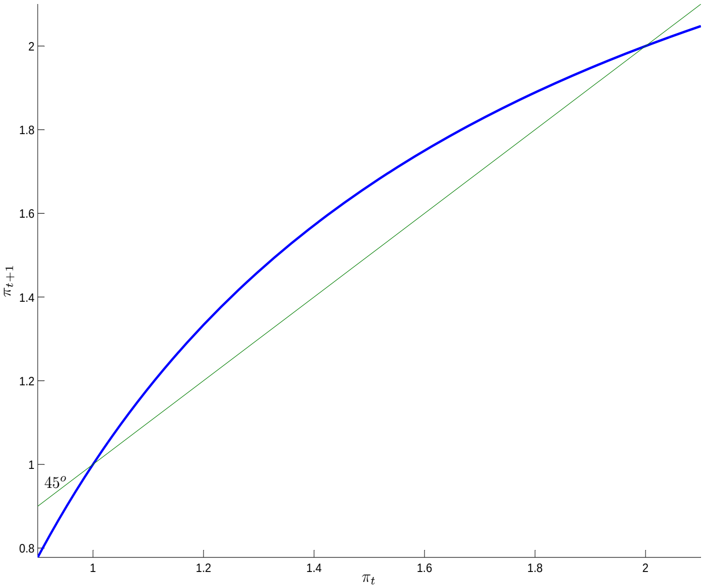
$$\pi = 3 - \frac{2}{\pi}$$

$\Longrightarrow$ there are two steady states

  - $\pi = 1$ (no inflation) and perfect consumption smoothing
  - $\pi = 2$ (high inflation) and no consumption smoothing at all

- Initial value for π_t not given, but given an initial condition the time path is fully determined

Rational expectations and stability



Rational expectations and stability

$$\begin{aligned}\pi_1 &: \text{ value in period 1}\\ \pi_1 &< 1: \text{ divergence}\\ \pi_1 &= 1: \text{ economy stays at low-inflation steady state}\\ \pi_1 &> 1: \text{ convergence to high-inflation steady state}\end{aligned}$$

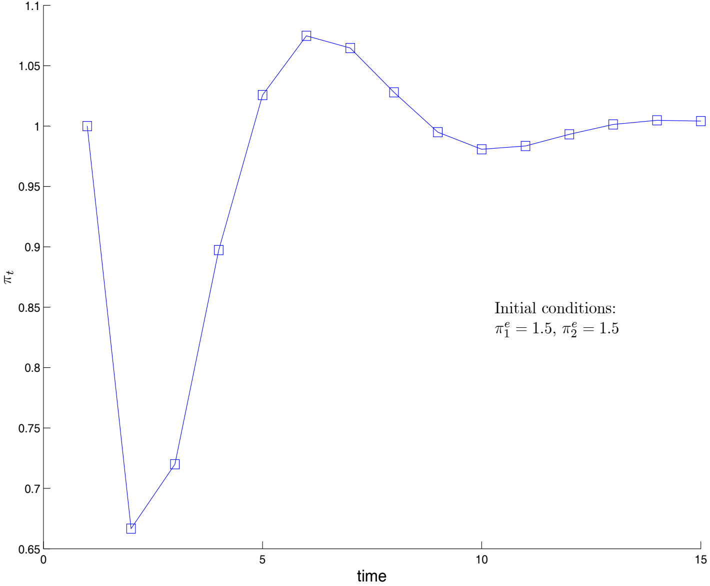
Alternative expecations

- Suppose that

$$\pi^e_{t+1} = \frac{1}{2}\pi_{t-1} + \frac{1}{2}\pi^e_t$$

- still the same two steady states, but
    - $\pi = 1$ is stable
    - $\pi = 2$ is not stable

Adaptive expectations and stability



Learning without feedback

Setup:

1. Agents know the complete model, except they do **not** know *dgp* exogenous processes
2. Agents use observations to update beliefs
3. Exogenous processes do not depend on beliefs
   $\Longrightarrow$ no feedback from learning to behavior of variable being forecasted

Learning without feedback & convergence

- If agents can learn the *dgp* of the exogenous processes, then you typically converge to REE
- They may not learn the correct *dgp* if
  - Agents use limited amount of data
  - Agents use misspecified time series process

Learning without feedback - Example

- Consider the following asset pricing model

$$P_t = \mathsf{E}_t \left[\beta \left(P_{t+1} + D_{t+1} \right) \right]$$

- If

$$\lim_{j \longrightarrow \infty} \beta^{t+j} D_{t+j} = 0$$

then

$$P_t = \mathsf{E}_t \left[\sum_{j=1}^{\infty} \beta^j D_{t+j} \right]$$

Learning without feedback - Example

- Suppose that

$$D_t = \rho D_{t-1} + \varepsilon_t,\ \varepsilon_t \sim N(0, \sigma^2) \qquad (1)$$

- REE:

$$P_t = \frac{D_t}{1 - \beta\rho}$$

(note that P_t could be negative so P_t is like a deviation from steady state level)

Learning without feedback - Example

- Suppose that agents do not know value of ρ
- Approach here:
  - If period t belief $= \widehat{\rho}_t$, then

$$P_t = \frac{D_t}{1 - \beta \widehat{\rho}_t}$$

  - Agents ignore that their beliefs may change,
    - i.e., $\widehat{\mathsf{E}}_t \left[P_{t+j} \right] = \mathsf{E}_t \left[\frac{D_{t+j}}{1-\beta\widehat{\rho}_{t+j}} \right]$ is assumed to equal $\frac{1}{1-\beta\widehat{\rho}_t} \mathsf{E}_t \left[D_{t+j} \right]$

Learning without feedback - Example

How to learn about ρ?

- Least squares learning using $\{D_t\}_{t=1}^{T}$ & correct *dgp*
- Least squares learning using $\{D_t\}_{t=1}^{T}$ & incorrect *dgp*
- Least squares learning using $\{D_t\}_{t=T-\bar{T}}^{T}$ & correct *dgp*
- Least squares learning using $\{D_t\}_{t=T-\bar{T}}^{T}$ & incorrect *dgp*
- Bayesian updating (also called rational learning)
- Lots of other possibilities

Convergence again

- Suppose that the true *dgp* is given by

$$
\begin{aligned}
D_t &= \rho_t D_{t-1} + \varepsilon_t \\
\rho_t &\in \left\{\rho_{\text{low}}, \rho_{\text{high}}\right\} \\
\rho_{t+1} &= \begin{cases} \rho_{\text{high}} \text{ w.p. } p(\rho_t) \\ \rho_{\text{low}} \text{ w.p. } 1 - p(\rho_t) \end{cases}
\end{aligned}
$$

- Suppose that agents think the true *dgp* is given by

$$D_t = \rho D_{t-1} + \varepsilon_t$$

- $\Longrightarrow$ Agents will never learn
(see homework for importance of sample used to estimate ρ)

Recursive least-squares

- time-series model:

$$y_t = x_t' \gamma + u_t$$

- least-squares estimator

$$\widehat{\gamma}_T = R_T^{-1} X_T' Y_t$$

where

$$\begin{aligned} X_T' &= \begin{bmatrix} x_1 & x_2 & \cdots & x_T \end{bmatrix} \\ Y_T' &= \begin{bmatrix} y_1 & y_2 & \cdots & y_T \end{bmatrix} \\ R_T &= X_T' X_T \end{aligned}$$

Recursive least-squares

$$
\begin{aligned}
R_T &= R_{T-1} + \frac{(x_T x_T' - R_{T-1})}{T} \\
\widehat{\gamma}_T &= \widehat{\gamma}_{T-1} + \frac{R_T^{-1} x_T \left(y_T - x_T' \widehat{\gamma}_{T-1}\right)}{T}
\end{aligned}
$$

Proof for R

$$\frac{X_T'X_T}{T} \stackrel{?}{=} \frac{X_{T-1}'X_{T-1}}{(T-1)} + \frac{x_Tx_T'}{T} - \frac{X_{T-1}'X_{T-1}}{T(T-1)}$$

$$\left(\frac{T-1}{T}\right) X_T'X_T \stackrel{?}{=} X_{T-1}'X_{T-1} + \frac{T-1}{T}x_Tx_T' - \frac{X_{T-1}'X_{T-1}}{T}$$

$$X_T'X_T - \frac{X_T'X_T}{T} \stackrel{?}{=} X_{T-1}'X_{T-1} + x_Tx_T' - \frac{x_Tx_T'}{T} - \frac{X_{T-1}'X_{T-1}}{T}$$

$$X_{T-1}'X_{T-1} + x_Tx_T' - \frac{X_{T-1}'X_{T-1}+x_Tx_T'}{T} \stackrel{?}{=} X_{T-1}'X_{T-1} + x_Tx_T' - \frac{x_Tx_T'+X_{T-1}'X_{T-1}}{T}$$

Proof for gamma

$$\left(X_T'X_T\right)^{-1} \times X_T'Y_T \stackrel{?}{=} \left(X_T'X_T\right)^{-1}\left(\begin{array}{c}\left(X_{T-1}'X_{T-1}\right)^{-1}X_{T-1}'Y_{T-1}+ \\ x_Ty_T \\ -x_Tx_T'\left(X_{T-1}'X_{T-1}\right)^{-1}X_{T-1}'Y_{T-1}\end{array}\right)$$

$$X_T'Y_T \stackrel{?}{=} \left(X_{T-1}'X_{T-1}+x_Tx_T'\right)\left(X_{T-1}'X_{T-1}\right)^{-1}X_{T-1}'Y_{T-1} + \left(\begin{array}{c} x_Ty_T \\ -x_Tx_T'\left(X_{T-1}'X_{T-1}\right)^{-1}X_{T-1}'Y_{T-1}\end{array}\right)$$

$$X_T'Y_T \stackrel{?}{=} \left(I+x_Tx_T'\left(X_{T-1}'X_{T-1}\right)^{-1}\right)X_{T-1}'Y_{T-1} + \left(\begin{array}{c} x_Ty_T \\ -x_Tx_T'\left(X_{T-1}'X_{T-1}\right)^{-1}X_{T-1}'Y_{T-1}\end{array}\right)$$

Reasons to adopt recursive formulation

- makes proving analytical results easier
- less computer intensive,
  - but standard LS gives the same answer
- there are intuitive generalizations:

$$\begin{aligned} R_T &= R_{T-1} + \omega(T)\left(x_T x_T' - R_{T-1}\right) \\ \widehat{\gamma}_T &= \widehat{\gamma}_{T-1} + \omega(T) R_T^{-1} x_T \left(y_T - x_T' \widehat{\gamma}_{T-1}\right) \end{aligned}$$

$\omega(T)$ is the "gain"

Learning with feedback

1. Explanation of the idea
2. Adaptive learning
   - E-stability and convergence
3. Least-squares learning
   - E-stability and convergence
4. Bayesian versus least-squares learning
5. Decision theoretic foundation of Adam & Marcet

Learning with feedback - basic setup

Model:

$$p_t = \rho \widehat{\mathsf{E}}_{t-1}[p_t] + \delta x_{t-1} + \varepsilon_t$$

RE solution:

$$\begin{aligned} p_t &= \frac{\delta}{1-\rho} x_{t-1} + \varepsilon_t \\ &= a_{\text{re}} x_{t-1} + \varepsilon_t \end{aligned}$$

What is behind model

Model:

$$p_t = \rho \widehat{\mathsf{E}}_{t-1}\left[p_t\right] + \delta x_{t-1} + \varepsilon_t$$

Stories:

- Lucas aggregate supply model
- Muth market model

See Evans and Honkapohja (2009) for details

Learning with feedback - basic setup

Perceived law of motion (PLM) at $t-1$:

$$p_t = \widehat{a}_{t-1} x_{t-1} + \varepsilon_t$$

Actual law of motion (ALM):

$$p_t = \rho \widehat{a}_{t-1} x_{t-1} + \delta x_{t-1} + \varepsilon_t = \left(\rho \widehat{a}_{t-1} + \delta \right) x_{t-1} + \varepsilon_t$$

Updating beliefs I: Adaptive

$$\text{ALM: } p_t = (\rho\widehat{a}_{t-1} + \delta)\, x_{t-1} + \varepsilon_t$$

Adaptive learning:

- $\widehat{a}_t = \rho\widehat{a}_{t-1} + \delta$
- could be rationalized if
  - agents observe x_{t-1} and ε_t
  - t is more like an iteration and in each iteration agents get (long) time-series to update

Adaptive learning: Convergence

$$\widehat{a}_t = \rho \widehat{a}_{t-1} + \delta$$

or in general

$$\widehat{a}_t - \widehat{a}_{t-1} = T\left(\widehat{a}_{t-1}\right)$$

Key questions:

1. Does $\widehat{a}_t$ converge?
2. If yes, does it converge to a_{RE}

Adaptive learning: E-stability

$$\widehat{a}_t - \widehat{a}_{t-1} = T\left(\widehat{a}_{t-1}\right)$$

Limiting behavior can be analyzed using

$$\frac{da}{d\tau} = T(a(\tau))$$

A solution a^* (e.g. a_{RE}) is *E-stable* if $T(a)$ is stable at a^*

E-stability

- $T(a)$ is stable if real part of the eigenvalues is negative:
  - $da > 0$ if $a < 0$
  - $da < 0$ if $a > 0$
- Here:

$$T(a) = (\rho - 1)\,a + \delta$$

$\implies$ convergence if $\rho - 1 < 0$

Adaptive learning: weakness

$$\text{ALM: } p_t = \left(\rho\widehat{a}_{t-1} + \delta\right) x_{t-1} + \varepsilon_t$$

- Agents (typically) do not observe $\left(\rho\widehat{a}_{t-1} + \delta\right)$
- $\Longrightarrow$ so not too convincing to set $\widehat{a}_t = \rho\widehat{a}_{t-1} + \delta$

Updating beliefs: LS learning

Suppose agents use least-squares learning

$$
\begin{aligned}
\widehat{a}_t &= \widehat{a}_{t-1} + \frac{R_t^{-1} x_{t-1} \left(p_t - x_{t-1} \widehat{a}_{t-1} \right)}{t} \\
&= \widehat{a}_{t-1} + \frac{R_t^{-1} x_{t-1} \left(\left(\rho \widehat{a}_{t-1} + \delta \right) x_{t-1} + \varepsilon_t - x_{t-1} \widehat{a}_{t-1} \right)}{t} \\
R_t &= R_{t-1} + \frac{\left(x_{t-1} x_{t-1} - R_{t-1} \right)}{t}
\end{aligned}
$$

Updating beliefs: LS learning

$$\begin{aligned}
\widehat{a}_t &= \widehat{a}_{t-1} + \frac{1}{t} R_t^{-1} x_{t-1} \left(p_t - x_{t-1}\widehat{a}_{t-1}\right) \\
&= \widehat{a}_{t-1} + \frac{1}{t} R_t^{-1} x_{t-1} \left(\left(\rho\widehat{a}_{t-1} + \delta\right) x_{t-1} + \varepsilon_t - x_{t-1}\widehat{a}_{t-1}\right) \\
R_t &= R_{t-1} + \frac{1}{t}\left(x_{t-1}x_{t-1} - R_{t-1}\right)
\end{aligned}$$

To get system with only lags on RHS, let $R_t = S_{t-1}$

$$\begin{aligned}
\widehat{a}_t &= \widehat{a}_{t-1} + \frac{1}{t} S_{t-1}^{-1} x_{t-1} \left(\left(\rho\widehat{a}_{t-1} + \delta\right) x_{t-1} + \varepsilon_t - x_{t-1}\widehat{a}_{t-1}\right) \\
S_t &= S_{t-1} + \frac{1}{t}\left(x_t x_t - S_{t-1}\right)\frac{t}{t+1}
\end{aligned}$$

Updating beliefs: LS learning

This can be written as

$$\widehat{\theta}_t = \widehat{\theta}_{t-1} + \frac{1}{t} Q(\widehat{\theta}_{t-1}, x_t, x_{t-1}, \varepsilon_t) = T(\widehat{\theta}_{t-1}, x_t, x_{t-1}, \varepsilon_t, t)$$

Key question

- If

$$\widehat{\theta}_t = T(\widehat{\theta}_{t-1}, x_t, x_{t-1}, \varepsilon_t, t)$$

then what can we "expect": about $\widehat{\theta}_t$?

- In particular, can we "expect" that

$$\lim_{t \longrightarrow \infty} \widehat{a}_t = a_{\text{re}}$$

Corresponding differential equation

Much can be learned from following differential equation

$$\frac{d\theta}{d\tau} = h\left(\theta\left(\tau\right)\right)$$

where

$$h\left(\theta\right) = \lim_{t\to\infty} \mathsf{E}\left[Q(\widehat{\theta}_{t-1}, x_t, x_{t-1}, \varepsilon_t)\right]$$

Corresponding differential equation

In our example

$$\begin{aligned} h(\theta) &= \lim_{t\to\infty} \mathsf{E}\left[Q(\theta, x_t, x_{t-1}, \varepsilon_t)\right] \\ &= \lim_{t\to\infty} \mathsf{E}\left[\begin{array}{c} S^{-1}x_{t-1}\left(\left(\rho a+\delta\right)x_{t-1}+\varepsilon_t - x_{t-1}a\right) \\ \left(x_t x_t - S\right)\frac{t}{t+1} \end{array}\right] \\ &= \left[\begin{array}{c} MS^{-1}\left(\left(\rho-1\right)a+\delta\right) \\ M-S \end{array}\right] \end{aligned}$$

where

$$M = \lim_{t\to\infty} \mathsf{E}\left[x_t^2\right]$$

Analyze the differential equation

$$\frac{d\theta}{d\tau} = \begin{bmatrix} MS^{-1}\left(\left(\rho - 1\right)a + \delta\right) \\ M - S \end{bmatrix}$$

$$\frac{d\theta}{d\tau} = 0 \text{ if } M = S \ \& \ a = \frac{\delta}{1 - \rho}$$

Thus, the (unique) rest point of $h\left(\theta\right)$ is the rational expectations solution

Implications of E-stability?

- Adaptive learning: no stochastics in $T(\cdot)$ mapping
- Recursive least-squares: stochastics in $T(\cdot)$ mapping
  - $\Longrightarrow$ what will happen is less certain, even if with E-stability

Implications of E-stability?

- If a solution **is not** E-stable:
    - $\Longrightarrow$ non-convergence is a probability 1 event
- If a solution **is** E-stable:
    - the presence of stochastics make the theorems non-trivial
    - we only have info about *mean dynamics*

Mean dynamics

See Evans and Honkapohja textbook for formal results.

- Theorems are a bit tricky, but are of the following kind:
  If a solution f^* is E-stable, then the time path under learning will either leave the neighborhood in finite time or will converge towards f^*. Moreover, the longer it does not leave this neighborhood, the smaller the probability that it will

- So there are two parts
  - mean dynamics: convergence towards fixed point
  - escape dynamics: (large) shocks may push you away from fixed point

Importance of Gain

$$\widehat{\gamma}_T = \widehat{\gamma}_{T-1} + \omega(T) R_T^{-1} x_T \left(y_T - x_T' \widehat{\gamma}_{T-1} \right)$$

- Gain in least squares updating formula, $\omega(T)$, plays a key role in theorems
- $\omega(T) \longrightarrow 0$ too fast: you may end up in somthing that is not an equilibrium
- $\omega(T) \longrightarrow 0$ too slowly:,you may not converge towards it
- So depending on the application, you may need conditions like

$$\sum_{t=1}^{\infty} \omega(t)^2 < \infty \text{ and } \sum_{t=1}^{\infty} \omega(t) = \infty$$

Bayesian learning

- LS learning has some disadvantages:
  - why "least-squares" and not something else?
  - how to choose gain?
  - why don't agents incorporate that beliefs change?
- Beliefs are updated each period
  $\Longrightarrow$ Bayesian learning is an obvious thing to consider

Bayesian versus LS learning

- LS learning $\neq$ Bayesian learning with uninformed prior
  at least not always

- Bullard and Suda (2009) provide following nice example

Bayesian versus LS learning

Model:

$$p_t = \rho_L p_{t-1} + \rho_0 \widehat{\mathsf{E}}_{t-1}\left[p_t\right] + \rho_1 \widehat{\mathsf{E}}_{t-1}\left[p_{t+1}\right] + \varepsilon_t \quad (2)$$

- Key difference with earlier model:
  - two extra terms

Bayesian versus LS learning

Solution:

- $$p_t = bp_{t-1} + \varepsilon_t$$

  where a is a solution to

  $$b = \rho_L + \rho_0 b + \rho_1 b^2$$

Bayesian learning - setup

- PLM:

$$p_t = \widehat{b}_{t-1} p_{t-1} + \varepsilon_t$$

  and ε_t has a known distribution

- plug PLM into (2) $\Longrightarrow$ ALM
  - but a Bayesian learner is a bit more careful

Bayesian learner understands he is learning

$$\begin{aligned}
\widehat{\mathsf{E}}_{t-1}\left[p_{t+1}\right] &= \widehat{\mathsf{E}}_{t-1}\left[\rho_L p_t + \rho_0 \widehat{\mathsf{E}}_t\left[p_{t+1}\right] + \rho_1 \widehat{\mathsf{E}}_t\left[p_{t+2}\right]\right] \\
&= \rho_L \widehat{b}_{t-1} p_{t-1} + \widehat{\mathsf{E}}_{t-1}\left[\rho_0 \widehat{\mathsf{E}}_t\left[p_{t+1}\right] + \rho_1 \widehat{\mathsf{E}}_t\left[p_{t+2}\right]\right] \\
&= \rho_L \widehat{b}_{t-1} p_{t-1} + \widehat{\mathsf{E}}_{t-1}\left[\rho_0 \widehat{b}_t p_t + \rho_1 \widehat{b}_t p_{t+1}\right]
\end{aligned}$$

- $\widehat{b}_t$ and p_{t+1} are both affected by ε_{t+1}!

Bayesian learner understands he is learning

- Bayesian learner realizes that

$$\widehat{\mathsf{E}}_{t-1}\left[\widehat{b}_t p_{t+1}\right] \neq \widehat{\mathsf{E}}_{t-1}\left[p_{t+1}\right]\widehat{\mathsf{E}}_{t-1}\left[p_{t+1}\right]$$

and calculates $\widehat{\mathsf{E}}_{t-1}\left[\widehat{b}_t p_{t+1}\right]$ explicitly

- LS learner forms expectations thinking that

$$\begin{aligned}\widehat{\mathsf{E}}_{t-1}\left[\widehat{b}_t p_{t+1}\right] &= \widehat{\mathsf{E}}_{t-1}\left[\widehat{b}_{t-1} p_{t+1}\right] \\ &= \widehat{b}_{t-1}\widehat{\mathsf{E}}_{t-1}\left[\left(\rho_L p + \rho_0 \widehat{b}_{t-1} + \rho_1 \widehat{b}_{t-1}\right) p_t\right]\end{aligned}$$

Bayesian versus LS learning

- Bayesian learner cares about a covariance term
- Bullard and Suda (2009) show that Bayesian is simillar to LS learning in terms of E-stability
- Such covariance terms more important in nonlinear frameworks
- Unfortunately not much done with nonlinear models

Learning what?

Model:

$$P_t = \beta \mathsf{E}_t \left[P_{t+1} + D_{t+1} \right]$$

- Learning can be incorporated in many ways.
- Obvious choices here:
  1. learn about *dgp* D_t and use true mapping for $P_t = P\left(D_t\right)$
  2. know *dgp* D_t and learn about $P_t = P\left(D_t\right)$
  3. learn about both

Learning what?

1. Adam, Marcet, Nicolini (2009): one can solve several asset pricing puzzles using a simple model if learning is learning about $\mathsf{E}_t\left[P_{t+1}\right]$ (instead of learning about *dgp* D_t)

2. Adam and Marcet (2011): provide micro foundations that this is a sensible choice

Simple model

Model:

$$P_t = \beta \mathsf{E}_t \left[P_{t+1} + D_{t+1} \right]$$

$$\frac{D_{t+1}}{D_t} = a\varepsilon_{t+1}$$

with

$$\mathsf{E}_t \left[\varepsilon_{t+1} \right] = 1$$

ε_t i.i.d.

Model properties REE

- Solution:

$$P_t = \frac{\beta a}{1 - \beta a} D_t$$

- P_t / D_t is constant
- P_t / P_{t-1} is i.i.d.

Adam, Marcet, & Nicolini 2009

PLM:

$$\widehat{\mathsf{E}}_t\left[\frac{P_{t+1}}{P_t}\right] = \gamma_t$$

ALM:

$$\begin{aligned} \frac{P_t}{P_{t-1}} &= \frac{1-\beta\gamma_{t-1}}{1-\beta\gamma_t} a\varepsilon_t = \left(a + \frac{a\beta\Delta\gamma_t}{1-\beta\gamma_t}\right)\varepsilon_t \\ \gamma_{t+1} &= \left(a + \frac{a\beta\Delta\gamma_t}{1-\beta\gamma_t}\right) \end{aligned}$$

Model properties with learning

- Solution is quite nonlinear
  - especially if γ_t is close to β^{-1}
- Serial correlation.
  - in fact there is momentum. For example:

$$\begin{aligned}\gamma_t &= a \;\&\; \Delta\gamma_t > 0 \Longrightarrow \Delta\gamma_{t+1} > 0 \\ \gamma_t &= a \;\&\; \Delta\gamma_t < 0 \Longrightarrow \Delta\gamma_{t+1} < 0\end{aligned}$$

- P_t/D_t is time varying

Adam, Marcet, & Nicolini 2011

Agent i does following optimization problem

$$\max \widehat{\mathsf{E}}_{i,t}\left[\cdot\right]$$

- $\widehat{\mathsf{E}}_{i,t}$ is based on a sensible probability measure
- $\widehat{\mathsf{E}}_{i,t}$ is not necessarily the true conditional expectation

Adam, Marcet, & Nicolini 2011

- Setup leads to standard first-order conditions but with $\widehat{\mathsf{E}}_{i,t}$ instead of E_t
- For example

$$P_t = \beta \widehat{\mathsf{E}}_{i,t} \left[P_{t+1} + D_{t+1} \right]$$

if agent i is not constrained

- Key idea:
    - price determination is difficult
    - agents do not know this mapping
    - $\Longrightarrow$ they forecast $\widehat{\mathsf{E}}_{i,t} \left[P_{t+1} \right]$ directly
    - $\Longrightarrow$ law of iterated expectations cannot be used because next period agent i may be constrained in which case the equality does not hold

Topics - Overview

1. E-stability and sun spots
2. Learning and nonlinearities
   Parameterized expectations
3. Two representations of sun spots

E-stability and sunspots

Model:

$$x_t = \rho \mathsf{E}_t\left[x_{t+1}\right]$$

x_t cannot explode

no initial condition

Solution:

$$\begin{aligned} |\rho| &< 1 : x_t = 0\ \forall t \\ |\rho| &\geq 1 : x_t = \rho^{-1} x_{t-1} + e_t\ \forall t \end{aligned}$$

where e_t is the sun spot (which has $\mathsf{E}_t\left[e_t\right] = 0$

Adaptive learning

PLM:

$$x_t = \widehat{a}_t x_{t-1} + e_t$$

ALM:

$$\begin{aligned} x_t &= \widehat{a}_t \rho x_{t-1} \\ &\Longrightarrow \widehat{a}_{t+1} = \widehat{a}_t \rho \end{aligned}$$

- thus divergence when $|\rho| > 1$ (sun spot solutions)

Adaptive learning

PLM:

$$x_t = \widehat{a}_t x_{t-1} + e_t$$

ALM:

$$\begin{aligned} x_t &= \widehat{a}_t \rho x_{t-1} \\ &\Longrightarrow \widehat{a}_{t+1} = \widehat{a}_t \rho \end{aligned}$$

- thus divergence when $|\rho| > 1$ (sun spot solutions)

Stability puzzle

- There are few counter examples and not too clear why sun spots are not learnable in RBC-type models
  - Sun spot solutions are learnable in some New Keynesian models (Evans and McGough 2005)
  - McGough, Meng, and Xue 2011 provide a counterexample and show that an RBC model with *negative* externalities has learnable sun spot solutions

PEA and learning

- Learning is usually done in linear frameworks
- PEA parameterized the conditional expectations in nonlinear frameworks
- $\Longrightarrow$ PEA is a natural setting to do both
    - adaptive learning
    - recursive learning

Model

$$P_t = \mathsf{E}\left[\beta\left(\frac{D_{t+1}}{D_t}\right)^{-\nu}(P_{t+1}+D_{t+1})\right] = G(X_t)$$

$$X_t : \text{state variables}$$

Conventional PEA in a nutshell

- Start with a guess for $G\left(X_t\right)$, say $g(x_t;\eta_0)$
  - $g\left(\cdot\right)$ may have wrong functional form
  - x_t may only be a subset of X_t
  - η_0 are the coefficients of $g\left(\cdot\right)$

Conventional PEA in a nutshell

- Iterate to find fixed point for η_i
  1. use η_i to generate time path $\{P_t\}_{t=1}^{T}$
  2. let
  $$\hat{\eta}_i = \arg\min_{\eta} \sum_t \left(y_{t+1} - g\left(x_t; \eta\right)\right)^2$$
  where
  $$y_{t+1} = \beta \left(\frac{D_{t+1}}{D_t}\right)^{-\nu} \left(P_{t+1} + D_{t+1}\right)$$
  3. Dampen if necessary
  $$\eta_{i+1} = \omega \hat{\eta}_i + \left(1 - \omega\right) \eta_i$$

Interpretation of conventional PEA

- Agents have beliefs
- Agents get to observe long sample generated with these beliefs
- Agents update beliefs
- Corresponds to adaptive expectations
  - no stochastics if T is large enough

Recursive PEA

- Agents form expectations using $g\left(x_t;\eta_t\right)$
- Solve for P_t using
$$P_t = g\left(x_t;\eta_t\right)$$
- Update beliefs using this one additional observation
- Go to the next period using η_{t+1}

Recursive methods and convergence

Look at recursive formulation of LS:

$$\widehat{\gamma}_t = \widehat{\gamma}_{t-1} + \frac{1}{t} R_t^{-1} x_t \left(y_t - x_t' \widehat{\gamma}_{t-1} \right)$$

- !!! $\Delta \hat{\gamma}_t$ gets smaller as t gets bigger

General form versus common factor represenation

Sun spot literature distinquishes between:

1. General form representation of a sun spot
2. Common factor representation of a sun spot

First consider non-sun-spot indeterminacy

Model:

$$\begin{aligned} k_{t+1} + a_1 k_t + a_2 k_{t-1} &= 0 \text{ or} \\ (1 - \lambda_1 L)(1 - \lambda_2 L) k_{t+1} &= 0 \end{aligned}$$

Also:

- k_0 given
- k_t has to remain finite

Multiplicity

Model:

$$k_{t+1} + a_1 k_t + a_2 k_{t-1} = 0 \quad \text{or}$$

$$(1 - \lambda_1 L)(1 - \lambda_2 L) k_{t+1} = 0$$

Also:

- k_0 given
- k_t has to remain finite

Multiplicity

Solution:

$$
\begin{aligned}
k_t &= b_1\lambda_1^t + b_2\lambda_2^t \\
k_0 &= b_1 + b_2
\end{aligned}
$$

Thus many possible choices for b_1 and b_2 if $|\lambda_1| < 1$ and $|\lambda_1| < 1$

Multiplicity

- What if we impose recursivity?

$$k_t = \bar{d} k_{t-1}$$

- Does that get rid of multiplicity? No, but it does reduce the number of solutions from ∞ to 2

$$\begin{aligned} \left(\bar{d}^2 + a_1 \bar{d} + a_2\right) k_{t-1} &= 0 \ \forall t \\ &\Longrightarrow \\ \left(\bar{d}^2 + a_1 \bar{d} + a_2\right) &= 0 \end{aligned}$$

the two solutions correspond to setting either λ_1 or λ_2 equal to 0

Back to sun spots

Doing the same trick with sun spots gives a solution with following two properties:

1. it has a *serially correlated* sun spot component with the same factor as the endogenous variable (i.e. the common factor)
2. there are two of these

General form representation

Model:

$$\begin{aligned} \mathsf{E}_t \left[k_{t+1} + a_1 k_t + a_2 k_{t-1} \right] &= 0 \quad \text{or} \\ (1 - \lambda_1 L)(1 - \lambda_2 L) k_{t+1} &= 0 \end{aligned}$$

General form representation:

$$\begin{aligned} k_t &= b_1 \lambda_1^t + b_2 \lambda_2^t + e_t \\ k_0 &= b_1 + b_2 + e_0 \end{aligned}$$

where e_t is serially uncorrelated

Common factor representation

Model:

$$\begin{aligned} \mathsf{E}_t\left[k_{t+1}+a_1k_t+a_2k_{t-1}\right] &= 0 \quad \text{or} \\ \left(1-\lambda_1L\right)\left(1-\lambda_2L\right)k_{t+1} &= 0 \end{aligned}$$

Common factor representation:

$$\begin{aligned} k_t &= b_1\lambda_i^t+\zeta_t \\ \zeta_t &= \lambda_i\zeta_{t-1}+e_t \\ k_0 &= b_i+\zeta_0 \\ \lambda_i &\in \{\lambda_1,\lambda_2\} \end{aligned}$$

where e_t is serially uncorrelated

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