GMM & Standard Errors for Business Cycle Statistics

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Overview

- Formulation of GMM estimation problem
  - large class of problems
- Formula for standard errors
- Calculating HAC estimator (correcting for heteroskedasticity and autocorrelation)

GMM problem

Underlying true model:

$$\mathsf{E}\left[h(x_t;\theta)\right] = 0_p$$

- $\theta : m \times 1$ vector with parameters
- $x : n \times 1$ vector of observables
- $h(\cdot) : p \times 1$ vector-valued function with $p \geq m$
- $0_p : p \times 1$ vector with zeros

Examples

OLS:

$$y_t = a + bx_t + u_t$$

$$\mathrm{E}\begin{bmatrix} u_t \\ u_t x_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- IV:

$$y_t = a + bx_t + u_t$$

$$\mathrm{E}\begin{bmatrix} u_t \\ u_t z_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Examples

DSGE:

$$u_{t+1} = \beta \left(1 + r_{t+1}\right) c_{t+1}^{-\gamma} - c_t^{-\gamma}$$

$$\mathsf{E}\left[u_{t+1} z_t\right] = 0_p$$

where z_t is a vector with variables in information set in period t

GMM estimation

$$\widehat{\theta}_T = \arg\min_{\theta \in \Theta} g(\theta; Y_T)' W_T g(\theta; Y_T)$$

where

$$\begin{aligned} g(\theta; Y_T) &= \sum_{t=1}^{T} h(x_t; \theta) / T \\ W_T &: p \times p \text{ weighting matrix} \\ Y_T &: \text{data} \end{aligned}$$

No weighting matrix needed if $p = m$

Asymptotic standard errors

$$W = \operatorname*{plim}_{T \longrightarrow \infty} W_T$$

$$D = \operatorname*{plim}_{T \longrightarrow \infty} \left. \frac{\partial g\left(\theta; Y_T\right)}{\partial \theta'} \right|_{\theta = \theta_0}$$

$$\theta_0 : \text{true } \theta$$

Asymptotic standard errors

$$
\begin{aligned}
T^{\frac{1}{2}}\left(\widehat{\theta}_T - \theta_0\right) &\longrightarrow N(0, V) \\
V &= \left(DWD'\right)^{-1} DW\Sigma_0 W'D' \left(DWD'\right)^{-1}
\end{aligned}
$$

Can be made easier

1. $p = m$ (no overidentifying restrictions)

$$T^{\frac{1}{2}}\left(\widehat{\theta}_T - \theta_0\right) \longrightarrow N(0, V)$$
$$V = \left(D\Sigma_0^{-1}D'\right)^{-1}$$

2. using optimal weighting matrix $W = \Sigma_0$

$$T^{\frac{1}{2}}\left(\widehat{\theta}_T - \theta_0\right) \longrightarrow N(0, V)$$
$$V = \left(D\Sigma_0^{-1}D'\right)^{-1}$$

What is left to figure out?

- "only" one thing left: estimate Σ_0
- Σ_0 is the variance-covariance matrix of $g(\theta; Y_T) = \sum_{t=1}^{T} h\left(x_t; \theta\right) / T$

Estimate variance of a mean

- $g(\theta; Y_T)$ is the mean of $h(x_t; \theta)$
- variance of $g(\theta; Y_T)$ is easy to calculate if $h\left(x_t; \theta\right)$ is serially uncorrelated

Estimate variance of a mean

General formula

$$\Sigma_0 = \sum_{j=-\infty}^{\infty} \mathsf{E}\left[h\left(x_t;\theta_0\right) h\left(x_{t-j};\theta_0\right)'\right]$$

How to estimate this?

$$\widehat{\Sigma}_0 = \sum_{j=-J}^{J} \kappa(j) \frac{\sum_{t=\max\{1,j+1\}}^{\min\{T,T+j\}} \left[h\left(x_t;\theta_0\right) h\left(x_{t-j};\theta_0\right)'\right]}{T}$$

Choice of kernel

- Truncated

$$\kappa(j) = \begin{matrix} 1 \text{ if } |j| \leq J \\ 0 \text{ o.w.} \end{matrix}$$

  - disadvantage: answer not necessarily psd

- Newey-West

$$\kappa(j) = 1 - j/J + 1$$

  - disadvantage: adds bias ($\kappa(j)$ should be $= 1$ for all j)

What to do in practice

1. Find $\widehat{\theta}_T$ (analytically or with minimization routine)
   - if $p > m$ first use non-optimal choice for W
2. Calculate $h(x_t;\widehat{\theta}_T)$ and estimate Σ_0
3. Estimate D (using $\widehat{\theta}_T$ for θ_0)
4. Calculate variance of $\widehat{\theta}_T$ using $\left(D\Sigma_0^{-1}D'\right)^{-1}$

Business cycles statistics

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$$\psi = \frac{\text{standard deviation } (c_t)}{\text{standard deviation } (y_t)}$$

❷

$$\rho = \text{correlation } (c_t, y_t)$$

Statistics easy to estimate, but what is the standard error?

Business cycles statistics

GMM problem for ψ

$$
\begin{aligned}
0 &= \mathsf{E}\left[c_t - \mu_c\right] \\
0 &= \mathsf{E}\left[y_t - \mu_y\right] \\
0 &= \mathsf{E}\left[\left(y_t - \mu_y\right)^2 \psi^2 - \left(c_t - \mu_c\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
D' &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2\mathsf{E}\left[\left(y_t - u_y\right)\right] & 2\mathsf{E}\left[\left(y_t - u_y\right)\psi^2\right] & 2\mathsf{E}\left[\left(y_t - u_y\right)^2 \psi\right] \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1\mathsf{E}\left[\left(y_t - u_y\right)^2 \psi\right] \end{bmatrix}
\end{aligned}
$$

Business cycles statistics

GMM problem for ρ

$$
\begin{aligned}
0 &= \mathrm{E}\left[c_t - \mu_c\right] \\
0 &= \mathrm{E}\left[y_t - \mu_y\right] \\
0 &= \mathrm{E}\left[\left(y_t - \mu_y\right)^2 \psi^2 - \left(c_t - \mu_c\right)^2\right] \\
0 &= \mathrm{E}\left[\left(y_t - \mu_y\right)^2 \psi\rho^2 - \left(c_t - \mu_c\right)\left(y_t - \mu_y\right)\right]
\end{aligned}
$$

$$
D' = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 2\mathrm{E}\left[\left(y_t - u_y\right)^2 \psi\right] & 0 \\
0 & 0 & \mathrm{E}\left[\left(y_t - u_y\right)^2 \rho^2\right] & 2\mathrm{E}\left[\left(y_t - u_y\right)^2 \psi\rho\right]
\end{bmatrix}
$$

References

- Den Haan, W.J., and A. Levin, 1997, A practioner's guide to robust covariance matrix estimation
  - a detailed survey of all the ways to estimate Σ_0. There are a lot more ways than those discussed here