Introduction to Numerical Methods

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"D", "S", & "GE"

- Dynamic
- Stochastic
- General Equilibrium

What is missing in the abbreviation?

- DSGE models include some form of *forward looking* behavior
  - Typically rational expectations
- Does forward looking makes these models difficult to solve?
  - Yes. How the economy behaves today depends on how agents think the economy will behave in the future for all possible outcomes
  - Agent-based models in which agents predict according to some of pre-specified rules are easier to solve

Recursive problems

Recursive problem

- same *state variables* $\Longrightarrow$ same choices
- Numerous problems are *not* recursive
- Some non-recursive problems can be rewritten as a recursive model by adding state variables
    - e.g., recursive contracts of Marcet & Marrimon

State variables are ...

- the variables that determine the outcomes in period t
    - predetermined variables like the capital stock
    - realizations of exogenous variables, like the productivity shock
    - *not* other endogenous variables to be determined within the period like prices
- Not always trivial to know what the state variables are

Example economy

$$\max_{\{c_{t+j},k_{t+j}\}_{j=0}^{\infty}} \mathsf{E}_t \sum_{j=0}^{\infty} \beta^j \frac{c_{t+j}^{1-\nu}-1}{1-\nu}$$

s.t.

$$c_{t+j} + k_{t+j} = z_{t+j}k_{t+j-1}^{\alpha} + (1-\delta)k_{t+j-1}$$
$$z_{t+j} = (1-\rho) + \rho z_{t+j-1} + \varepsilon_t$$
$$k_{t-1} \text{ given}$$

$$\mathsf{E}_{t+j}[\varepsilon_{t+j+1}] = 0 \;\&\; \mathsf{E}_{t+j}[\varepsilon_{t+j+1}^2] = \sigma^2$$

Alternative notation

$$c_t + k_t = z_t k_{t-1}^{\alpha} + (1-\delta) k_{t-1}$$

- k_t is the *end*-of-period t capital stock, chosen in t and productive in period $t+1$
- It is also possible that k_t stands for the *beginning*-of-period t capital stock. Then the budget constraint is written as

$$c_t + k_{t+1} = z_t k_t^{\alpha} + (1-\delta) k_t$$

- This is just a change in notation, not in the model

Bellman equation

- If the problem is recursive it can be rewritten using the Bellman equation

$$v(k_{-1}, z) = \max_{c,k} \frac{c^{1-\nu} - 1}{1-\nu} + \mathsf{E}\left[\beta v(k, z_{+1})\right]$$

$$\text{s.t.}$$

$$c + k = z k_{-1}^{\alpha} + (1-\delta) k_{-1}$$

First-order conditions

$$
\begin{aligned}
c + k &= z k_{-1}^{\alpha} + (1-\delta) k_{-1} \\
c^{-\nu} &= \beta \mathsf{E}\left[c_{+1}^{-\nu} \left(\alpha z_{+1} k^{\alpha-1} + 1 - \delta \right) \right]
\end{aligned}
$$

Solution have the following form:

$$\begin{aligned} c &= c(k_{-1}, z) \\ k &= k(k_{-1}, z) \end{aligned}$$

- Why is it difficult to find these solutions?

What should the solutions satisfy?

$$c(k_{-1},z) + k(k_{-1},z) = zk_{-1}^{\alpha} + (1-\delta)k_{-1}$$

$$c(k_{-1},z)^{-\nu} = \beta \mathsf{E}\left[c(k(k_{-1},z),z_{+1})^{-\nu}\left(\alpha z_{+1}k(k_{-1},z)^{\alpha-1} + 1 - \delta\right)\right]$$

with

$$z_{+1} = (1-\rho) + \rho z + \varepsilon_{+1}$$

Example with analytical solution

- If $\delta = \nu = 1$ then we know the analytical solution. It is

$$k_t = \alpha\beta \exp(z_t) k_{t-1}^{a}$$
$$c_t = (1-\alpha\beta) \exp(z_t) k_{t-1}^{a}$$

or

$$\begin{aligned} \ln k_t &= \ln(\alpha\beta) + \alpha \ln k_{t-1} + z_t \\ \ln c_t &= \ln(1-\alpha\beta) + \alpha \ln k_{t-1} + z_t \end{aligned}$$

Different ways to numerically solve these models

1. Methods that focus on the first-order conditions
   1. projection methods
   2. perturbation methods
2. Methods that are based on the Bellman equation
   - similar to projection methods
   - often slower, but can handle more complex models (e.g. discontinuities)

References

- Den Haan, W.J., Dynamic optimization problems (available online)
  - basic stuff on getting first-order conditions, transversality conditions, state variables, and dynamic programming
- Den Haan, W.J., Equilibrium models (available online)
  - continuation but now using equilibrium infinite-horizon (mainly monetary models) and OLG models
- Ljungqvist, L. and T.J. Sargent, 2004, Recursive Macroeconomic Theory
  - describe many DSGE models and their properties
- Stockey, N.L, R.E. Lucas Jr, with E.C. Prescott, 1989, Recursive methods in economic dynamics