Solving the incomplete markets model with aggregate uncertainty using parameterized cross-sectional distributions

Yann ALGAN[*], Olivier ALLAIS,[†] and Wouter J. DEN HAAN[‡§]

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Abstract

This note describes how the incomplete markets model with aggregate uncertainty in Den Haan, Judd, and Juillard (2009) is solved using standard quadrature and projection methods. This is made possible by linking the aggregate state variables to a parameterized density that describes the cross-sectional distribution. A simulation procedure is used to find the best shape of the density within the class of approximating densities considered. This note compares several simulation procedures in which there is—as in the model—no cross-sectional sampling variation.

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[*]Sciences Po, OFCE, Paris, France. E-mail: yann.algan@ens.fr.

[†]INRA, UR1303 ALISS, F-94205 Ivry-sur-Seine, France. E-mail: Olivier.Allais@ivry.inra.fr.

[‡]Corresponding author: Department of Economics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands and CEPR. E-mail: wdenhaan@uva.nl. Phone: +31 20 5255237. Fax: +31 20 5254254.

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1 Introduction

This paper describes the algorithm used to solve the model with incomplete markets and aggregate risk of Den Haan, Judd, and Juillard (2009). The algorithm of Krusell and Smith (1998), the most popular algorithm to solve this type of model, consists of an iterative procedure and in each iteration a simulation of the economy with the approximating solution is used to solve for the law of motion of aggregate capital. The simulation procedure of Krusell and Smith (1998) has two types of sampling variation. The first is due to using a finite instead of a continuum of agents. As shown below, this sampling variation can be avoided. The sampling variation that is due to the aggregate shock, however, seems unavoidable. Using simulated data to obtain numerical solutions has two disadvantages. First, by introducing sampling noise the policy functions themselves become stochastic. This effect can be reduced by using long time series, but sampling noise disappears at a slow rate. Second—and more importantly—the values of the state variables used to find the best fit for the aggregate law of motion are endogenous and are typically clustered around their means. But accuracy can be improved by using values that are more spread out.[1] In particular, the numerical literature advocates the use of Chebyshev nodes to ensure uniform convergence and the procedure used here allows for this efficient choice of grid points.

The algorithm described here uses projection methods and can—in principle—solve the model without relying on any simulation procedure. Using projection procedures to solve a model with a continuum of agents typically requires a parameterization of the cross-sectional distribution as in Den Haan (1997).[2] We improve on the procedure proposed in Den Haan (1997) in the following way. If one parameterizes the cross-sectional distribution, then all parameters of the density are state variables. For example, if one use a Normal density then there are two parameters, i.e., the mean and the variance, and thus two state variables.[3] But note that using a Normal density has implications for the higher-order moments. These implied higher-order moments may not be correct. For example, a Normal density implies no skewness, but the model one tries to solve may have a skewed distribution. In that case one could allow for more general approximating functions with more free

[1] Recall that the standard errors of regression coefficients, $\sigma^2(X'X)^{-1}$, are lower when the x-values are more spread out.

[2] Den Haan and Rendahl (2009) show that aggregation *without* explicit distributional assumptions is possible when the individual policy functions are linear in the coefficients. They implicitly obtain information about the distribution by approximating auxiliary policy rules.

[3] As shown below, one can establish a mapping between the parameters of the approximating density and a set of moments even if more flexible densities are used. Instead of using the parameters of the density, we always use moments as state variables.

parameters. The problem of adding coefficients to the approximating density is that one also adds state variables. Our procedure uses an approximation for the density that allows for more flexibility, but does *not* increase the number of state variables.

The idea is the following. Suppose one starts with the Normal as the approximating cross-sectional density and uses the mean and the variance as state variables. Using this approximating density one can obtain a numerical solution of the model using standard projection methods and without any simulation. Now that one has obtained a numerical solution one can ask the question whether the cross-sectional density is described accurately with a Normal density. To answer this question one has to rely on a simulation. Suppose that after simulating a panel and calculating the higher-order (unconditional) cross-sectional moments, one concludes that the Normal does not provide an accurate representation. When using the algorithm of Den Haan (1997), one would use a higher-order approximation of the cross-sectional distribution and increase the number of state variables.

But one can also modify the functional form of the cross-sectional distribution without adding state variables. This is the approach followed here, that is, the information obtained from the simulation is used to modify the functional form of the cross-sectional distribution. Thus, if the Normal is not accurate one would use at each point on the grid a density that (i) implies values for the higher-order moments equal to the values found in the simulation and, of course, (ii) implies values for the lower-order moments that are included as state variables. The algorithm iterates on this procedure until the information provided by the simulation is consistent with the assumptions made about the shape of the cross-sectional distribution. The philosophy that underlies our algorithm is similar to the one in Reiter (2009). The differences are mainly in terms of implementation, which is less cumbersome for our algorithm.

Although we rely on a simulation procedure, it plays a much smaller role than in, for example, the algorithm of Krusell and Smith (1998); it is only used to determine the shape of the density. The procedure to solve for the policy rules uses standard projection techniques without a simulation step.

Algan, Allais, and Den Haan (2008) (AAD hereafter) propose a new procedure to simulate cross-sections with a continuum of agents. The most common procedure to simulate models with a continuum of agents consists of using a finite number of agents and a random number generator to draw the idiosyncratic shocks. Consequently, the results are subject to cross-sectional sampling variation. Models with a continuum of agents do not have this property and most solution procedures are based on this lack of sampling variation.[4] AAD show that sampling variation can

[4]For example, solution procedures typically specify that next period's distribution is fully determined by the current distribution and aggregate shocks.

be substantial and that properties of the laws of motion may be overlooked because of the presence of cross-sectional noise. In this note, we compare three procedures that all avoid cross-sectional sampling variation.

2 Algorithm

This section provides an overview of the key ingredients of the algorithm.[5]

Projection method. The numerical solution of the incomplete markets model with aggregate uncertainty in Den Haan, Judd, and Juillard (2009) consists of a policy function $k'(\varepsilon, k, a, s; \Psi_k)$, where ε is the (exogenous) individual employment status,[6] k the individual capital stock, a the exogenous aggregate state, s a set of variables that characterizes the cross-sectional joint distribution of capital and employment status, and Ψ_k the coefficients of the policy function. The variable s refers to the beginning-of-period distribution after the new employment status has been observed.

The standard projection procedure to solve for Ψ_k consists of the following three steps.

1. Construct a grid of the state variables.

2. At each grid point, define an error term, v, given values for ε, k, a, and s as

$$
\begin{aligned}
v(\varepsilon,k,a,s;\Psi_k) &= \tfrac{1}{c} - \textstyle\sum_{\varepsilon',a'} \left[\frac{\beta(r'+1-\delta)}{c'}\right] \pi_{aa'\varepsilon\varepsilon'} \\
&= \frac{1}{(r+1-\delta)k+wl-k'(\varepsilon,k,a,s;\Psi)} \\
&\quad - \textstyle\sum_{\varepsilon',a'} \left[\frac{\beta(r'+1-\delta)}{(r'+1-\delta)k'(\varepsilon,k,a,s;\Psi)+w'l'-k'(\varepsilon',k',a',s';\Psi)}\right] \pi_{aa'\varepsilon\varepsilon'} \\
&= \frac{1}{(r+1-\delta)k+wl-k'(\varepsilon,k,a,s;\Psi)} \\
&\quad - \textstyle\sum_{\varepsilon',a'} \left[\frac{\beta(r'+1-\delta)}{(r'+1-\delta)k'(\varepsilon,k,a,s;\Psi)+w'l'-k'(\varepsilon',k'(\varepsilon,k,a,s;\Psi),a',s';\Psi)}\right] \pi_{aa'\varepsilon\varepsilon'} \\
&\text{with} \\
l = (1-\tau)\bar{l}\varepsilon + \mu(1-\varepsilon),\ & l' = (1-\tau')\bar{l}\varepsilon' + \mu(1-\varepsilon'), \\
r = \alpha a \left(\tfrac{K}{\bar{l}(1-u(a))}\right)^{\alpha-1},\ & r' = \alpha a' \left(\tfrac{K'}{\bar{l}(1-u(a'))}\right)^{\alpha-1}, \\
w = (1-\alpha)a \left(\tfrac{K}{\bar{l}(1-u(a))}\right)^{\alpha},\ & w' = (1-\alpha)a' \left(\tfrac{K'}{\bar{l}(1-u(a'))}\right)^{\alpha}, \\
\tau = \tfrac{\mu u(a)}{\bar{l}(1-u(a))}&, \text{ and } \tau' = \tfrac{\mu u(a')}{\bar{l}(1-u(a'))}.
\end{aligned}
\tag{1}
$$

[5] A more in depth discussion can be found in AAD.

[6] The value of ε is equal to 0 when the agent is unemployed and equal to 1 when the agent is employed.

Here, K is the aggregate capital stock, u is the unemployment rate (which is determined by the aggregate exogenous state a), r is the rental rate, and w is the wage rate. If the worker is employed then he works $\bar{l}$ hours and his labor income equals $(1-\tau)w\bar{l}$. If he is unemployed then he receives μw. The first-order conditions of the agent, evaluated using the numerical solution $v(\cdot|\psi)$, correspond to the following set of conditions:

$$\begin{array}{l} v(\varepsilon,k,a,s;\Psi_k) \geq 0, \\ v(\varepsilon,k,a,s;\Psi_k)k' = 0, \text{ and} \\ k' \geq 0, \end{array} \tag{2}$$

for all possible values of ε, k, a, and s.

3. Ψ_k is found by minimizing some objective criterion that weighs the values of the error terms at the nodes of the grid.

Two things are needed to be able to evaluate $v(\varepsilon,k,a,s;\Psi_k)$. First, s and a must pin down K. If K would be an element of s then this would be trivial. Second, it must be possible to obtain the values of s' as a function of a, a', and s. This can be done if s implies an actual cross-sectional distribution. The cross-sectional distribution of the current period together with the individual policy function can then determine the characteristics of next period's distribution (and thus s') using standard quadrature techniques. Next, we explain how this can be done.

Linking s to a cross-sectional distribution. Let the first $N_{\overline{M}}$ moments of the strictly positive capital holdings of agents with employment status ω be given by $\overleftarrow{m^{\omega,j}}$, with $j \in \{1,\cdots,N_{\overline{M}}\}$ and suppose that these are elements of s.[7] To link this set of moments with a density, we approximate the density of individual capital holdings with a flexible functional form $P(k,\rho^\omega)$ and choose the parameters ρ^ω such that the moments of the density coincide with those specified.[8] The following functional form is used.

$$P(k,\rho^\omega) = \rho_0^\omega \exp\left(\begin{array}{c} \rho_1^\omega\left[k-\overleftarrow{m^{\omega,1}}\right]+ \\ \rho_2^\omega\left[\left(k-\overleftarrow{m^{\omega,1}}\right)^2-\overleftarrow{m^{\omega,2}}\right]+\cdots+ \\ \rho_{N_{\overline{M}}}^\omega\left[\left(k-\overleftarrow{m^{\omega,1}}\right)^{N_{\overline{M}}}-\overleftarrow{m^{\omega,N_{\overline{M}}}}\right] \end{array}\right). \tag{3}$$

[7] We set ω equal to e when the agent is employed and equal to u when the agent is unemployed. An arrow pointing left (right) denotes beginning(end)-of-period values.

[8] To completely characterize the cross-sectional distribution one would also need to include in s the fraction of agents at the constraint.

The advantage of this particular functional form is that the coefficients $\rho_1^\omega, \cdots, \rho_{N_{\overline{M}}}^\omega$ can be found with the following minimization routine.

$$\min_{\rho_1^\omega, \rho_2^\omega, \cdots, \rho_{N_{\overline{M}}}^\omega} \int_0^\infty P(k, \rho^\omega) dk. \tag{4}$$

The reason is that the first-order conditions of this minimization problem are exactly the conditions that the first $N_{\overline{M}}$ moments are equal to the set of specified moments.

$$\begin{aligned}
\int_0^\infty \left[k - \overleftarrow{m^{\omega,1}}\right] \; P(k;\rho^\omega) dk &= 0 \\
\int_0^\infty \left[(k - \overleftarrow{m^{\omega,1}})^2 - \overleftarrow{m^{\omega,2}}\right] \; P(k;\rho^\omega) dk &= 0 \\
&\cdots \\
\int_0^\infty \left[(k - \overleftarrow{m^{\omega,1}})^{N_{\overline{M}}} - \overleftarrow{m^{\omega,N_{\overline{M}}}}\right] \; P(k;\rho^\omega) dk &= 0
\end{aligned} \tag{5}$$

AAD show that the minimization problem is convex, which means that the first-order conditions are monotone and thus easy to solve.[9] The coefficient ρ_0^w is determined by the condition that the density integrates to one. By increasing the number of moments one increases the order of the approximating polynomial and the accuracy of the approximation.

The approximating densities are used to determine s' and are not necessarily of interest to the researcher. In fact, it may very well be the case that accurate predictions of s' can be obtained with approximating densities that are not accurate in all aspects. We document this in Section 3 by showing that an approximating density with continuous support (for strictly postive capital levels) can accurately predict next period's moments even though the true cross-sectional density has points with positive point mass, i.e., the CDF is discountinous.

Solving the model without simulation. The algorithm as it is described now can be executed without any simulation. That is, ψ_k can be chosen to minimize a loss function over the residuals defined in equation 1. The problem is that to obtain an accurate solution one would need several moments as state variables, that

[9] For alternative specifications of the functional form one would have to solve the coefficients from a system like (5), which likely to be a more challenging numerical problem.

is, the value of $N_{\overline{M}}$ cannot be too low. This statement seems to contradict the well-known finding of Krusell and Smith (1998) that the cross-sectional mean is a sufficient state variable. But note that higher-order moments may not matter in predicting next period's prices for different reasons. The first is that changes in them truly have no effect. But they also may not matter because their time-series variation is low.[10] In the latter case, the effect of the higher-order moments would be captured by the constant term in the time-series regression that relates next period's mean capital stock to this period's mean capital stock. AAD find that higher-order moments do matter and that they have to be included to get the shape of the cross-sectional distribution right. But using information about higher-order moments to get the shape of the cross-sectional distribution right does not mean that one has to include all higher-order moments as state variables. This is the idea behind reference moments and will be discussed next.

Reference moments. In the algorithm described so far, the cross-sectional density at a node on the grid was determined by the set of moments included as state variables. But suppose that in addition to the moments that are included as state variables one also has information about higher-order moments. Higher-order moments that are not included as state variables, but used to determine the density are referred to as *reference moments*. For example, when only the mean is used as a state variable one may have information about the variance. But if the variance is not a state variable, then one needs to provide information about it from *outside* the projection procedure. One possibility would be to obtain this information from the solution of the model without aggregate uncertainty. Another possibility, and the one that is chosen here, is the following. Start with a guess for the reference moments, solve the model using the algorithm described above, and then simulate the economy. The simulated panel can be used to update the information about the reference moments. The simplest thing to do would be to use the unconditional values of the cross-sectional moments. We use the values of the cross-sectional moments conditional on the realization of a. Alternatively, one could relate the reference moments to the values of all included aggregate state variables. Note that this would only require a simple regression using data from the simulated economy. At each node on the grid, the regression results can then be used to determine the appropriate values of the reference moments.

[10] Another possibility is that the time-variation of higher-order moments is related to movements in the mean.

Discussion of choices made. Several choices were motivated by convenience, such as, similarity to choices made in other numerical work. Here we discuss choices that the reader should be aware off. To simplify the description of the algorithm, we assumed that we had an approximation function for next period's state variable, k. But one can just as well approximate the consumption choice or the conditional expectation and we chose the latter.[11] We approximate the conditional expectation using Chebyshev polynomials. This and a grid constructed using Chebyshev nodes leads to several desirable convergence properties.[12] But there are also disadvantages. First, the conditional expectation displays a sharp non-differentiability at the lowest level of k at which the agents chooses a zero capital stock, $\overline{k}(\varepsilon, a, s)$. For $k \leq \overline{k}(\varepsilon, a, s)$, however, the conditional expectation does not have to be approximated, so we simply approximate the conditional expectation on those grid points at which the constraint is not binding. But this means not using the full set of Chebyshev nodes and some of the optimality properties may be lost. Moreover, the conditional expectation has other—less pronounced—non-differentiabilities due to the interaction of the constraint and the discrete support of ε.[13] In this particular problem these disadvantages are minor because the constraint only binds at very low levels of k, but when the constraint plays a more important role the reader should seriously consider using splines.

Another choice that the reader should be aware of is that we parameterize the law of motion relating s' to the current-period aggregate state variables. Conditional on this law we then solve for the individual policy rules and then update the aggregate law of motion by projecting the calculated values of s' on the grid on the approximating functional form. It is possible that this sequential updating improves the stability of the algorithm when solving complex models. But if convergence is not an issue, then it makes more sense not to use this two-step procedure. Even if one would like an approximating aggregate law of motion, then it would be better to solve the model using the algorithm outlined above and then simply get an approximation for the aggregate law after one has obtained the solution of the model. Further information on the choices made can be found in the appendix and in AAD.

[11] Some motivation for choosing the conditional expectation is given in Christiano and Fischer (2000).

[12] See Judd (1998, p. 221).

[13] See Den Haan (1997, Figure 2).

3 Simulating a continuum cross-section of agents

Simulation procedures fulfill an important role in the numerical analysis of models with heterogeneous agent models. The popular procedure of Krusell and Smith (1998) uses simulated cross-sectional moments to determine the aggregate law of motion. Even in our algorithm—that is designed to obtain numerical solutions to the policy functions without simulation procedures—we still use a simulation procedure to reduce the dimension of the set of state variables while keeping an accurate shape of the cross-sectional density. And even if an algorithm does not rely on a simulation procedure at all, then many characteristics of the solution can only be determined using a simulation procedure.

Given the importance of simulation procedures, it is important to compare alternatives. The most popular procedure is to use a finite set of agents and to use a random number generator to determine the realizations of the idiosyncratic and common shocks. But this means that the outcome is subject to cross-sectional sampling variation, whereas both the model and the algorithm typically rely on there being none. AAD show that this sampling variation can be substantial especially for the smaller group in the population such as the unemployed.

There are, however, procedures that avoid cross-sectional sampling variation, but to the best of our knowledge these have not been compared. Section 3.1 outlines three different simulation procedures, Section 3.2 compares the three simulation procedures for the model discussed here and for a model in which the CDF displays substantial discontinuities.

3.1 Three simulation procedures

To simplify the exposition we explain how to simulate across time a cross-sectional distribution of capital holdings when there are no shocks and no constraint, that is, when the policy function for k' is given by $k(k)$.[14]

[14] For most sensible choices of $k(k)$, the distribution would then converge towards a single point. Adding stochastic elements that would prevent this is easy, but would make the exposition somewhat more tedious.

3.1.1 Simulation procedure of AAD

Let $f_t(k)$ be the distribution of capital holdings in period t and let f_1 be given.[15,16] Calculate the first $N_{\overline{M}}$ moments of the distribution of k' using quadrature methods. The inputs are the policy function, $k'(k)$, and the initial distribution, f_1. Using the procedure discussed in Section 2 one can then obtain the density $f_2(k)$ that corresponds to these $N_{\overline{M}}$ moments.[17] Iteration on this procedure gives a time series $f_t(k)$. Given $f_t(k)$, any characteristic of the cross-sectional distribution can be calculated.

3.1.2 Grid-based procedure of Young[18]

Construct a grid of capital holdings, $\kappa_{\bar{j}}$, $\bar{j} = 0, \cdots, N$, and let $p_t^{\bar{j}}$ be equal to the mass of agents with a capital stock equal to $\kappa_{\bar{j}}$. We have

$$\sum_{\bar{j}=0}^{N} p_t^{\bar{j}} = 1.$$

Calculate the values for p_{t+1}^{j} using the following algorithm.

- Initialize by setting $p_{t+1}^{j} = 0$ for all j.
- Calculate the values of p_{t+1}^{j} using the following procedure for $\bar{j} = 0, \cdots, N$.
  - Calculate $k'(\kappa_{\bar{j}})$. Let j be such that $\kappa_j \leq k'(\kappa_{\bar{j}}) < \kappa_{j+1}$.
  - The mass at the $\bar{j}^{\text{th}}$ grid point, $p_t^{\bar{j}}$, is allocated to the two grid points that enclose the choice $k'(\kappa_{\bar{j}})$—i.e., the j^{th} and the $(j+1)^{\text{th}}$ grid point—using the distance of $k'(\kappa_{\bar{j}})$ to the two grid points to determine the fractions. Thus,

$$p_{t+1}^{j} = p_{t+1}^{j} + \frac{\kappa_{j+1} - k'(\kappa_{\bar{j}})}{\kappa_{j+1} - \kappa_j} p_t^{\bar{j}}$$

[15] Alternatively, one can start the procedure with $N_{\overline{M}}$ moments. The density $f_1(k)$ can then be determined using the procedure of Section 2.

[16] It is easy to modify the procedure to include a constraint. $f_t(k)$ would in that case be the density of the strictly positive capital holdings and one would in addition keep track of the mass of agents at the constraint.

[17] If there are no constraints on the range of k, then one has to choose a lower and an upperbound for k that are outside the ergodic set or at least such that the mass below and above these two values is very small.

[18] Proposed in Young (2009).

and

$$p_{t+1}^{j+1} = p_{t+1}^{j+1} + \frac{k'(\kappa_{\bar{j}}) - \kappa_j}{\kappa_{j+1} - \kappa_j} p_t^{\bar{j}}$$

- The sum of all the p_{t+1}^js is by construction equal to 1.

In the model without aggregate uncertainty, this procedure can be expressed as a linear system that can be used to solve for the stationary distribution (and thus the equilibrium aggregate capital stock) by solving for the normalized eigenvector corresponding to the unit eigenvalue.

3.1.3 Grid-based procedure of Ríos-Rull[19]

Again construct a grid of capital holdings, κ_j, $j = 0, \cdots, N$. Let $\bar{p}_t^0$ be the mass of agents at κ_0 and let $\bar{p}_t^j$ be equal to the mass of agents with a capital stock bigger than κ_{j-1} and less than or equal to κ_j, for $j > 0$.[20] This mass is assumed to be distributed uniformly between grid points. We have

$$\sum_{j=0}^{N} \bar{p}_t^j = 1.$$

Let x^j be equal to the capital level at which an agent chooses κ_j.[21] Note that whereas the procedure proposed by Young simply uses the capital choice at a set of nodes, this procedure uses the inverse of the capital choice. Thus,

$$k'(x^j) = \kappa_j. \tag{6}$$

Now compute the distribution function of next period's capital at the grid points as

$$\bar{P}_{t+1}^j = \int_0^{x^j} d\bar{P}_{t+1}(k) = \sum_{j=0}^{\bar{j}} \bar{p}_t^j + \frac{x^j - \kappa_{\bar{j}}}{\kappa_{\bar{j}+1} - \kappa_{\bar{j}}} \bar{p}_t^{j+1}, \tag{7}$$

where $\bar{j} = \bar{j}(x^j)$ is the largest value of j such that $\kappa_j \leq x^j$. The second equality follows from the assumption that mass is distributed uniformly between grid points. Note that $\bar{p}_{t+1}^0 = \bar{P}_{t+1}^0$ and $\bar{p}_{t+1}^j = \bar{P}_{t+1}^j - \bar{P}_{t+1}^{j-1}$ for $j > 0$. Modifying the distribution to take into account unemployment risk is—as for the procedure of Young—straightforward.

[19] This procedure is used in Ríos-Rull (1997), Heathcote (2005), Reiter (2006), and Den Haan (2009).

[20] Note that $\bar{p}_t^j$ is not equal to p_t^j (used in the last subsection), except for $j = 0$. p_t^j is the mass *at* a grid point and $\bar{p}^j$ is the mass *between* grid points.

[21] Note that if the capital choice would depend on aggregate state variables then x would be time varying.

3.2 Comparison and discussion

3.2.1 Experiment 1

In this experiment, we use our numerical solution for the individual policy functions of the model outlined above to simulate the cross-sectional distribution across time with the three simulation procedures for 10,000 periods. The initial distribution is identical to the one used in Den Haan (2009).

We find that time-series plots of characteristics of the cross-sectional distributions, such as moments and percentiles, are very similar. Not surprisingly, the largest differences are observed in the description of the lower tail. For example, for the 1st percentile we find for the employed (unemployed) that the differences are 1.35% (2.48%), 1.60% (1.78%), and 0.76% (1.60%) for AAD versus Ríos-Rull, AAD versus Young, and Young versus Ríos-Rull, respectively.[22]

3.2.2 Experiment 2

Generating an accurate simulated panel for the model of the model presented here is relatively easy, because there are very few constrained agents, which means that any subsequent jumps in the CDF for higher levels of capital are very small. Moreover, the marginal propensity to save is almost constant and only varies with capital at low levels of capital.

Therefore, we also consider an example in which the marginal propensity to save varies strongly with capital and jumps in the CDF are important. Both features may give difficulties for the procedure of AAD. The continuous approximating density used in the AAD procedure, of course, misses the jumps of the CDF. Missing these jumps is not important as long as the marginal propensity to save is (locally) constant, but may matter if the savings function is nonlinear.

In the second experiment, the individual policy function, $k'(\varepsilon, k)$ is assumed to be equal to

$$k'(0, k) = \max\{0, k - 25\}$$

for the unemployed agent ($\varepsilon = 0$) and is equal to

$$k'(1, k) = -\gamma_0 + k + \exp(\alpha_0 + \alpha_1 k + \alpha_2 k^2)$$

for the employed agent ($\varepsilon = 1$). For the chosen parameter values,[23] the marginal propensity to save of an employed agent varies from 0 when $k = 0$ to almost 1 when

[22] Since the mass of agents in the first percentile is very small (between 3% and 9% for the unemployed), these percentage differences imply very small differences for the mass of agents in the first percentile.

[23] $\alpha_0 = 2.70805$, $\alpha_1 = -0.06667$, $\alpha_2 = 0.000326$, and $\gamma_0 = -0.6$.

$k = 99$. The laws of motion of the exogenous random variables are as in experiment 1.

Although these policy functions do not depend on the aggregate state, the choices still do because the employment status depends on the aggregate state. The chosen policy function may look strange, but is motivated by its ability to generate large jumps in the cross-sectional distribution. When the aggregate state randomly changes, then the distributions generated with different procedures look similar because of these random aggregate shocks. The differences between the solution procedures become more clear if we keep the economy in the same aggregate state. That is, the economy remains in either the good or the bad state. Those are the results reported here. The fraction of agents at the constraint is now substantially higher than in the problem discussed above. This higher fraction of constrained agents leads to several substantial jumps in the CDF as is clear in Figure 1 that plots the CDF obtained with the AAD and Young procedure when the economy has been in the bad state for a long time period.

Figure 1 documents that the CDFs obtained with the different procedures display substantial differences. The CDF generated with the Young procedure nicely displays the jumps in the CDF that are also present in the true CDF. The CDF generated by the AAD procedure, of course, doesn't have any jumps but it nicely approximates the distribution. That is, the inability of AAD to capture the jumps does not lead to a systematic bias.

This is also documented by the time series of standard characteristics of the cross-sectional distribution that are very similar across the two procedures. This is documented in figures 2 and 3, which plot the simulated mean capital stocks and the fraction of agents at the constraint when the economy is (and remains) in the bad aggregate state. Figures 4 and 5 report the results when the economy is (and remains) in the good aggregate state.

3.2.3 Discussion

Although the procedures are quite different, they generate very similar results in both experiments. Of course, our results may not carry over to all problems and one always should check whether the simulated data are accurate. But the results presented here indicate that convenience may be an important element in the choice made as well.

In terms of programming, the easiest procedure is the one proposed by Young (2009). In contrast to the grid-based procedure of Ríos-Rull (1997), it does not

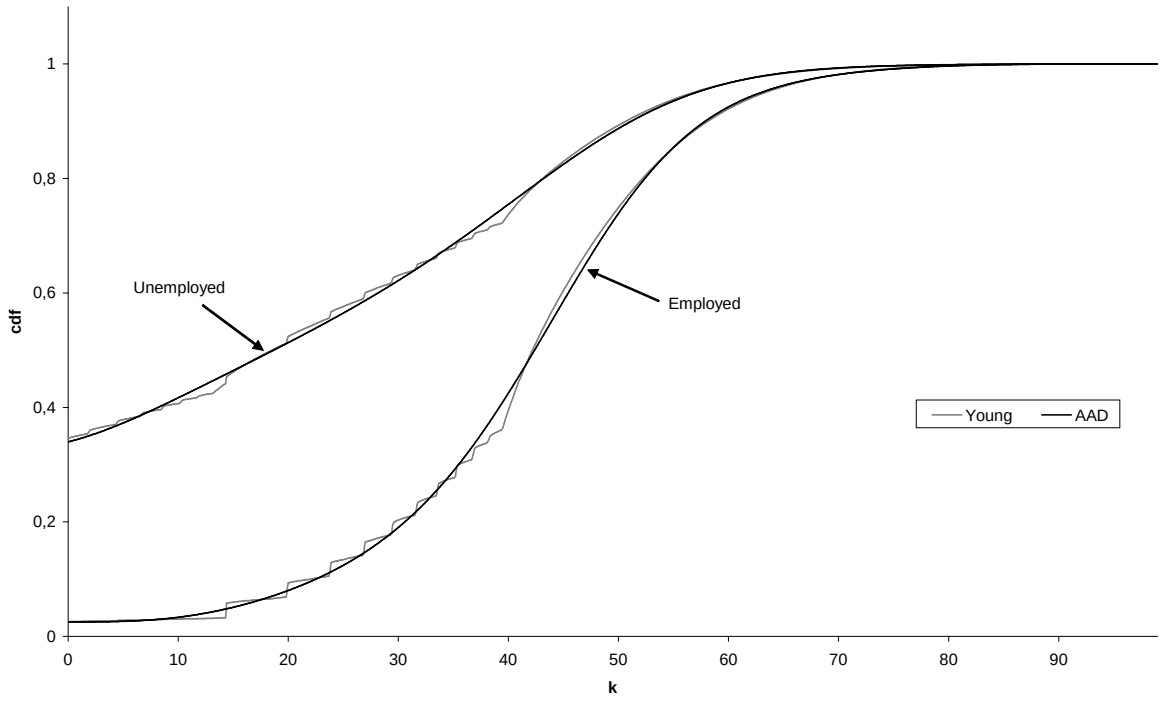

Figure 1: The cumulative distibution function for the employed and the unemployed when the economy has been in the bad state for a long time.

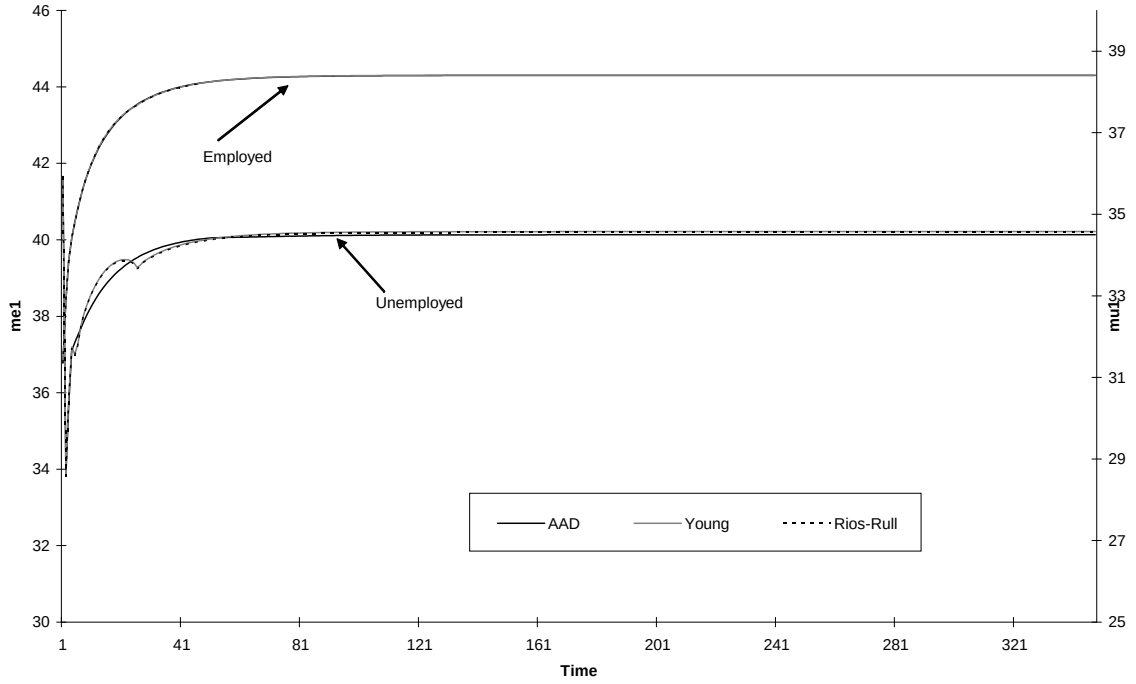

Figure 2: The means of the employed and unemployed when the economy is in the bad state

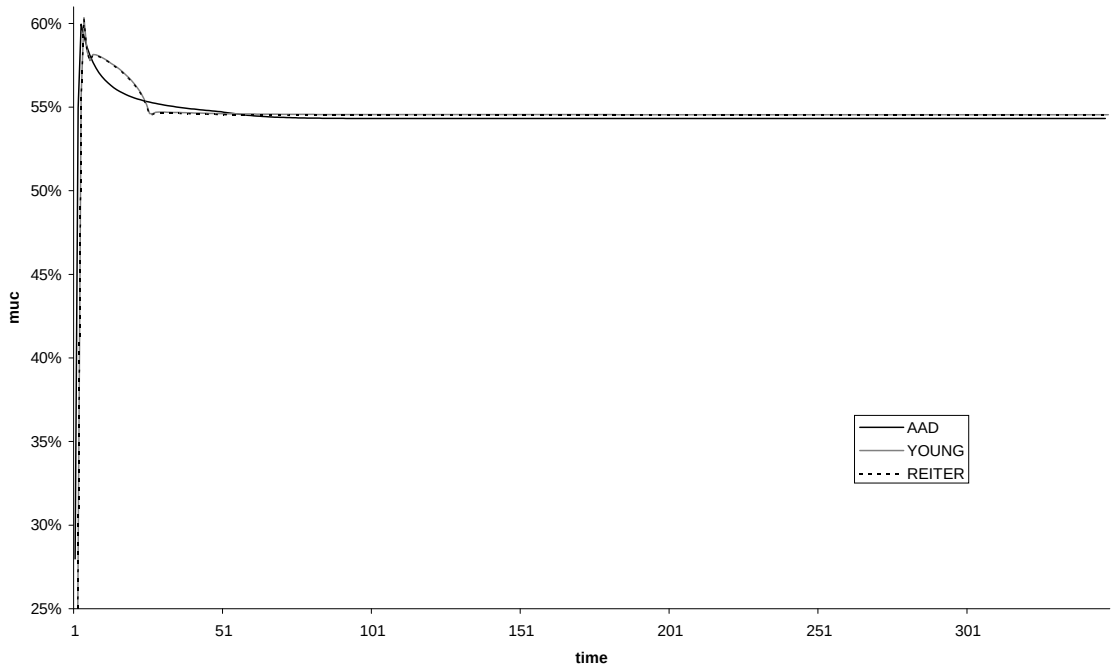

Figure 3: The fraction of agents at the constraint when the economy is in the bad state

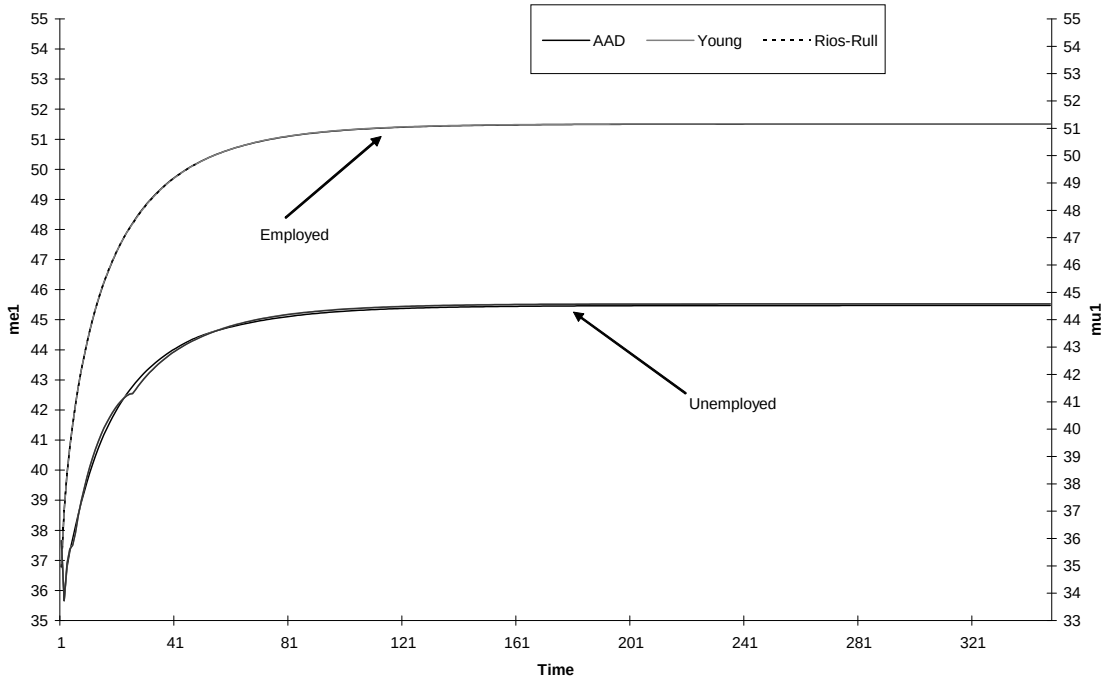

Figure 4: The means of the employed and unemployed when the economy is in the good state

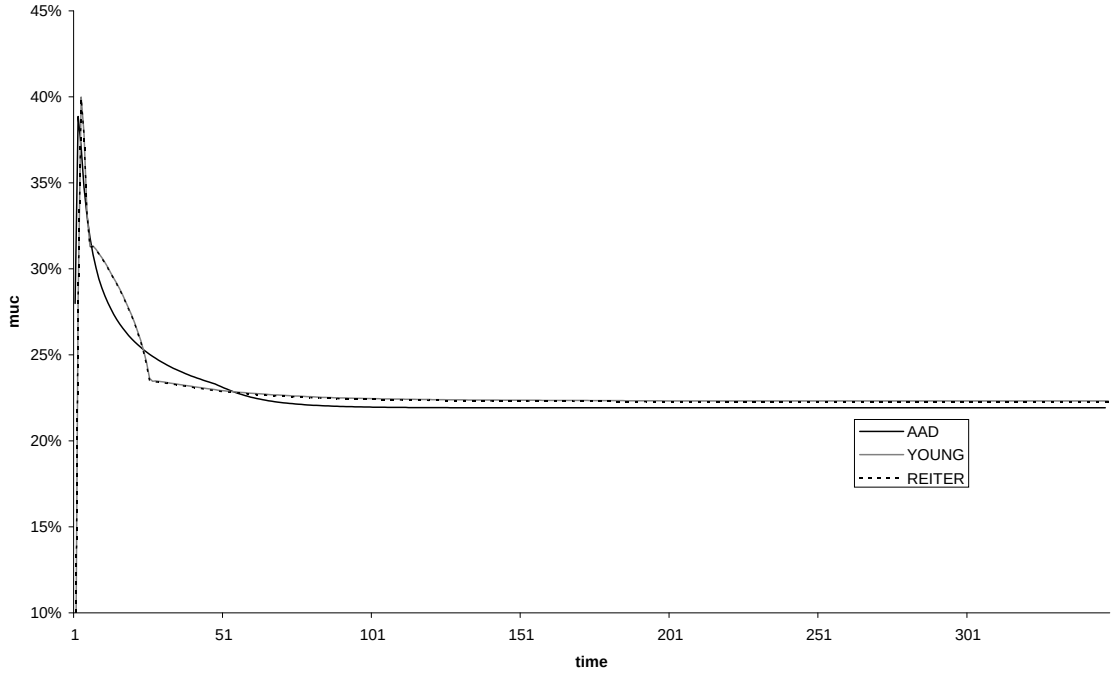

Figure 5: The fraction of agents at the constraint when the economy is in the good state

require calculating the inverse, which can be a costly operation.[24] Both grid procedures allow quite naturally for discontinuities in the CDF. But the second example showed that the procedure of AAD also can lead to an accurate characterization of the movements across time of key characteristics of the cross-sectional distribution, even in the presence of substantial discontinuities. The main advantage of the procedure of AAD is that it characterizes the cross-sectional distribution with a much smaller number of parameters. For the procedures discussed here, the simulation procedure of AAD uses ten parameters whereas the grid-based method uses one thousand. For some applications, it may be extremely helpful to limit the number of parameters.

A Appendix

Table 1: Parameters of the numerical procedure

variable	range	number of grid points	order of approximating polynomial
k	$[0, 99]$	50 grid points	27
$\overrightarrow{m_{-1}^{u,c}}$	$[0, 0.002]$	5 grid points	3
$\overleftarrow{m^{e,1}}$	$[35, 42.4]$	5 grid points	3
$\overleftarrow{m^{u,1}}$	$[33.5, 41.5]$	5 grid points	3

The state variables used are

$$s = \left[a_{-1}, a, \overrightarrow{m_{-1}^{u,c}}, \overleftarrow{m^{e,1}}, \overleftarrow{m^{u,1}}\right],$$

where $\overrightarrow{m_{-1}^{u,c}}$ stands for the fraction of constrained unemployed agents at the end of the last period, and $\overleftarrow{m^{\omega,1}}$ stands for the beginning-of-period mean capital holdings of agents with employment status ω and strictly positive capital holdings. Note that this set of state variables has enough information to determine $\overleftarrow{m^{u,c}}$ and $\overleftarrow{m^{e,c}}$.[25] In addition to these moments, we use five higher-order moments (for both the employed and the unemployed) to determine the density of the cross-sectional distribution. In the simulation we use a total of ten moments.

[24] Reiter (2006) proposes several approximating steps to speed up the procedure.

[25] Alternatively, we could have used $s = [a, \overleftarrow{m^{u,c}}, \overleftarrow{m^{e,c}}, \overleftarrow{m^{e,1}}, \overleftarrow{m^{u,1}}]$. The advantage of our choice is that a_{-1} can take on only two values and is, therefore, "cheaper" as a state variable than an additional fraction of constrained agents.

Parameter settings of the numerical procedure, such as the order of the polynomial and the number of grid points, are given in Table 1. We use Chebyshev nodes as the grid points and the indicated range of the state variable is used to transform the variable into one that is between -1 and 1. For the exogenous random variables we use two grid points related to the two possible realizations.

References

[1] Algan, Yann, Olivier Allais, and Wouter J. den Haan, 2008, Solving heterogeneous agent models with parameterized cross-sectional distributions, *Journal of Economic Dynamics and Control* 32, p. 875-908.

[2] Christiano, Lawrence J. and Jonas D. Fischer, 2000, Algorithms for solving dynamic models with occasionally binding constraints, *Journal of Economic Dynamics and Control* 24, p. 1179-1232.

[3] Den Haan, Wouter J., 1997, Solving dynamic models with aggregate shocks and heterogeneous agents, *Macroeconomic Dynamics* 1, p. 355-386.

[4] Den Haan, Wouter J., 2009, Comparison of solutions to the incomplete markets model with aggregate uncertainty, *Journal of Economic Dynamics and Control*, this issue.

[5] Den Haan, Wouter J., Kenneth L. Judd, and Michel Juillard, 2009, Computational suite of models with heterogenous agents: Incomplete Markets and Aggregate Uncertainty, *Journal of Economic Dynamics and Control,* this issue.

[6] Den Haan, Wouter J., and Pontus Rendahl, 2009, Solving the incomplete markets model with aggregate uncertainty using explicit aggregation, *Journal of Economic Dynamics and Control*, this issue.

[7] Heathcote, Jonathan, 2005, Fiscal policy with heterogeneous agents and incomplete markets, *Review of Economic Studies* 72, p. 161-188.

[8] Judd, Kenneth L., 1992, Projection methods for solving aggregate growth models, *Journal of Economic Theory* 58, p. 410-452.

[9] Judd, Kenneth L., 1998, *Numerical methods in economics*, MIT Press, Cambridge, MA.

[10] Krusell, Per and Smith, Anthony, 1998, Income and wealth heterogeneity in the macroeconomy, *Journal of Political Economy*, p. 868-896.

[11] Reiter, Michael, 2009, Solving heterogeneous agent models by projection and perturbation, *Journal of Economic Dynamics and Control*, forthcoming.

[12] Reiter, Michael, 2009, Solving the incomplete markets model with aggregate uncertainty by backward induction, *Journal of Economic Dynamics and Control*, this issue.

[13] Ríos-Rull, J. Victor, 1997, Computation of equilibria in heterogeneous agent models, Federal Reserve Bank of Minneapolis Staff Report 231, p. 238-264.

[14] Young, Eric, 2009, Solving the incomplete markets model with aggregate uncertainty using the Krusell-Smith algorithm and non-stochastic simulations, *Journal of Economic Dynamics and Control*, this issue.