Solving and Simulating Models with Heterogeneous Agents and Aggregate Uncertainty

Yann ALGAN, Olivier ALLAIS, Wouter J. DEN HAAN, and Pontus RENDAHL*

September 18, 2008

Abstract

This chapter reviews different algorithms to solve and simulate models with heterogeneous agents and aggregate uncertainty. This is a new area of research and several algorithms have only been recently developed. It is therefore surprising that many authors spend very little on documenting the accuracy of their solution, at best report weak accuracy measures such as the standard error of the regression equation and the $R^2$, and rely on aggregation results found in other environments.

Key Words: Incomplete Markets, Numerical Solutions, Projection Methods, Perturbation Methods, Parameterized Densities.

JEL Classification: C63,D52.

*Algan: Paris School of Economics and University Paris East; yann.algan@ens.fr. Allais: INRA, Corela; Olivier.Allais@ivry.inra.fr. Den Haan: University of Amsterdam and CEPR; wdenhaan@uva.nl. Rendahl: University of California at Davis; prendahl@ucdavis.edu The authors would like to thank Ken Judd, Michel Juillard, and Alex Ludwig, for useful comments.
1 Introduction

Computational algorithms to solve models with heterogeneous agents and aggregate uncertainty appeared in the second half of the nineties. Early examples are Den Haan (1996, 1997) and Krusell and Smith (1997, 1998). The presence of aggregate uncertainty means that the cross-sectional distribution of agents’ characteristics becomes a state variable. Thus, unless the number of agents is small, one has to solve for policy rules with a large set of arguments. In particular, in the popular framework with a continuum of agents, the set of state variables would be infinite dimensional. A common feature of existing algorithms is to summarize this infinite dimensional object with a limited set of statistics that summarize the distribution.

Krusell and Smith (1998) consider a model in which employment risk is not fully insurable because of borrowing constraints and missing markets. They show that in this environment the mean of the capital stock is a sufficient statistic. The reason for this important finding is that—except for the very poor—the marginal propensity to save is very similar across agents. Since there are not many poor and their wealth is small, the similar marginal propensities to save of the other agents implies that redistributions of wealth have no effect on aggregate savings and, thus, not on market prices. Important is that aggregate savings is endogenous. This implies that the average agent can build up wealth to diminish the chance of being poor and being affected by the borrowing constraint. Den Haan (1997) considers a model in which aggregate savings are zero. In this model the borrowing constraint is more frequently binding and higher-order moments do matter.

This chapter reviews the traditional algorithms and the newer ones that have been developed. It is important to consider alternative algorithms. The first generation of heterogeneous-agent models consists of relatively simple models in which, moreover, only one moment of the aggregate distribution seemed to matter. But models are getting more complex and it is not clear whether the result that only the mean matters will automatically carry over to other environments. Krusell and Smith (2006) say in their conclusion that "we foresee important examples of such phenomena [quantitatively convincing models
with large departures from aggregation] to be discovered in future research". The existing algorithms differ in important aspects from each other. While the first algorithms relied heavily on simulation procedures, the newer algorithms try to build algorithms using building blocks proven to be fruitful in the numerical literature such as projections methods and perturbation techniques.

In addition to reviewing algorithms, this chapter also reviews different procedures to simulate economies with a continuum of agents and discusses procedures to check for accuracy. A common procedure is to simulate using a large but finite number of agents. This introduces unnecessary sampling variation that may be substantial for some groups of the population. Given that there are alternative procedures that are more accurate and faster it doesn’t make sense to rely on stochastic simulation procedures.

An important topic of this chapter is a discussion on how to check for accuracy. The standard procedure to check for accuracy is to use the $R^2$ or the standard error of the regression, the two accuracy measures that Krusell and Smith (1998) focus on. Den Haan (2006) shows that these are very weak measures and aggregate laws that are very different from the true aggregate law can still have a very high $R^2$ and a low standard error. Den Haan (2006) also proposes an alternative accuracy test that is more powerful. Note that Krusell and Smith (1998) actually consider several alternative accuracy measures. One of them, the maximum forecast error at a long forecast horizon turns out to be also much more powerful in detecting inaccuracies than the $R^2$ and the standard error.

**Things to include in the introduction** Tricks to get rid of heterogeneity such as New Keynesian models and price dispersion or models with financial friction and linear technology

### 2 Example model

In this section, I review the model of Krusell and Smith (1998). This model has been used to compare the outcomes of different numerical algorithms. Its relative simplicity makes it easy to explain the key features of the different algorithms. Another reason to focus on
this model is that its aggregation properties have been quite influential.

**Problem for the individual agent.** The economy consists of a unit mass of ex ante identical households. Each period, agents face an idiosyncratic shock $e$ that determines whether they are employed, $e = 1$, or unemployed, $e = 0$. An employed agent earns an after-tax wage rate of $(1 - \tau_t)r_t^w$ and an unemployed agent receives unemployment benefits $\mu r_t^w$. Note that Krusell and Smith set $\mu$ equal to zero. This is the only difference with their model. Markets are incomplete and the only investment available is capital accumulation. The net rate of return on this investment is equal to $r_t^k - \delta$, where $r_t^k$ is the rental rate and $\delta$ is the depreciation rate. Agent’s $i$ maximization problem is as follows:

$$\max_{\{c_{i,t}, k_{i,t+1}\}} \sum_{t=0}^{\infty} \beta^t \ln(c_{i,t})$$

s.t. $c_{i,t} + k_{i,t+1} = r_t^k k_{i,t} + (1 - \tau_t)r_t^w e_{i,t} + \mu r_t^w (1 - e_{i,t}) + (1 - \delta) k_{i,t}$

$$k_{i,t+1} \geq 0$$

Here $c_{i,t}$ is the individual level of consumption, $k_{i,t}$ is the agent’s beginning-of-period capital, and $l$ is the time endowment. Note that log utility is chosen only to simplify the notation.

The Euler equation is given by

$$\frac{1}{c_{i,t}} = E_t \left[ \beta_r^{r_{t+1} + 1 - \delta} \frac{c_{i,t+1}}{c_{i,t+1}} \right]$$

**Firm problem.** Markets are competitive and the production technology of the firm is characterized by a Cobb-Douglas production function. Consequently, firm heterogeneity is not an issue. Let $K_t$ and $L_t$ stand for per capita capital and the employment rate, respectively. Per capita output is given by

$$Y_t = z_t K_t^\alpha (L_t)^{1-\alpha}$$

and prices by

$$r_t^w = (1 - \alpha) z_t \left( \frac{K_t}{L_t} \right)^{\alpha}$$

$$r_t^k = \alpha z_t \left( \frac{K_t}{L_t} \right)^{\alpha-1}$$
Aggregate productivity, $z_t$, is an exogenous stochastic process that can take on two values, $1 - \Delta^z$ and $1 + \Delta^z$.

**Government** The only role of the government is to tax employed agents and to redistribute funds to the unemployed. We assume that the government’s budget is balanced each period. This implies that the tax rate is equal to

$$\tau_t = \frac{\mu u_t}{L_t},$$

where $u_t = 1 - L_t$ denotes the unemployment rate in period $t$.

**Exogenous driving processes.** There are two stochastic driving processes. The first is aggregate productivity and the second is the employment status. Both are assumed to be first-order Markov processes. We let $\pi_{zz'}ee'$ stand for the probability that $z_{t+1} = z'$ and $e_{t+1} = e'$ when $z_t = z'$ and $e_t = e'$. These transition probabilities are chosen such that the unemployment rate can take on only two values. That is, $u_t = u^b$ when $z_t = z^b$ and $u_t = u^g$ when $z_t = z^g$ with $u^b > u^g$.

**Equilibrium** Krusell and Smith (1998) consider recursive equilibria in which the policy functions of the agent depend on his employment status, $\varepsilon_i$, his beginning-of-period capital holdings, $k_i$, aggregate productivity, $z$, and the cross-sectional distribution of capital holdings.\(^1\) An equilibrium then consists of the following.

- Individual policy functions that solve the agent’s maximization problem.
- A wage and a rental rate that are determined by 4 and 5, respectively.
- A transition law for the cross-sectional distribution of capital, that is consistent with the investment policy function. Let $f_t$ represent the beginning-of-period cross-sectional distribution of capital and the employment status after the employment

---

\(^1\)Miao (2006) shows the existence of a recursive equilibrium, but also uses expected payoffs as state variables. Krusell and Smith (2006) show existence using the smaller set of state variables used by Krusell and Smith (1998), but under an assumption that cannot be checked from primitives. It remains, therefore, not clear whether a recursive equilibrium exists when the smaller set of state variables is used. For a numerical solution this is less important in the sense that approximation typically entails not using all information.
status has been realized. The transition law can be written as

$$f_{t+1} = \Upsilon(z_{t+1}, z_t, f_t).$$

(7)

This law of motion reveals an advantage of working with a continuum of agents. Because we can apply a law of large numbers, we know that conditional on $z_{t+1}$ there is no uncertainty in determining $f_{t+1}$.

3 Algorithms - overview

There are now several algorithms to solve models with heterogeneous agents and aggregate uncertainty combining different tools from the numerical solution literature. They vary from the "pure" perturbation approach of Preston and Roca (2006) to the almost "pure" projection method approach of Algan, Allais, and Den Haan (2006). This section is split in two parts. Section 3.1 discusses procedures that rely on projection and simulation approaches. These are global procedures in the sense that properties at different parts of the state space affect the numerical solution. Section 3.2 discusses perturbation approaches in which the numerical solution is pinned down by the derivatives at one particular point. The purpose of this section is to explain—in a hopefully intuitive manner—the key aspects of the different algorithms. Section 3.1.2 contains a discussion of the different choices.

Simulations are used in several algorithms and typically play a role even when the algorithm itself does not rely on simulations. The reason is that many properties of the model can often only be calculated by using simulations. Since the model assumes a continuum of agents, simulation is nontrivial and it is not obvious what the best way to simulate is. Given the importance of this question, simulation procedures are discussed separately in Section 4 and not touched upon in this section.

3.1 Projection and simulation approaches

This section discusses three quite different approaches. It discusses the approach developed in Den Haan (1996), which is a pure simulations approach, the approach of Krusell and Smith (1998) in which simulations are used to determine the aggregate laws of motion, and
the approach of Algan, Allais, and Den Haan (2006), which is based mainly on projection methods.

### 3.1.1 Obtain aggregate policy functions from simulation

The most popular algorithm used in the literature is the one developed in Krusell and Smith (1998). Its underlying idea is as follows. As discussed above, we focus on equilibria in which (i) individual policy functions depend on $s_t$ and (ii) next period’s cross-sectional distribution is a time-invariant function of the current distribution and realizations of the aggregate shock. Krusell and Smith (1998) approximate the infinite-dimensional cross-sectional distribution with a finite set of moments, $m_t$. A numerical law of motion now consists of an individual policy function (as a function of the vector $[e_t, k_t, z_t, m_t]$) and a law of motion for $m_{t+1}$ of the form

$$m_{t+1} = \Gamma(z_{t+1}, z_t, m_t).$$

The idea underlying of the algorithm is fairly straightforward. Suppose that the law of motion $\Gamma(\cdot)$ is known. Then the problem of solving for the individual policy rules is standard and one can use any of the available algorithms. Note that in solving for the individual policy functions one will run into the problem of evaluating next period’s prices and thus next period’s aggregate capital stock, but this can be calculated using the mapping $\Gamma$. The algorithm then consists of the following iterative scheme

- Start with an initial guess for $\Gamma$, say $\Gamma_0$.

- Using this guess, solve for the individual policy rule.

- Construct a time series for $m_t$. That is, using the solution for the individual policy rule simulate the economy. Each period calculate the elements of $m_t$ from the cross-sectional distribution.

- Use a regression analysis to get a new estimate for the law of motion $\Gamma$. This is $\Gamma_1$.

- Iterate until $\Gamma_{j+1}$ is sufficiently close to $\Gamma_j$.  

\[\text{The aggregate capital stock is either an element of } m_t \text{ or can be calculated from } m_t \text{ (and possibly } z_t).\]
In Sections and , I will discuss in more detail how one decides whether one has included enough moments and how to evaluate accuracy.

### 3.1.2 Obtain aggregate and individual policy functions through simulation

Den Haan (1996) also assumes that the cross-sectional distribution is characterized by a finite set of moments, \( m_t \). He solves for the individual policy rules from a simulation procedure, which avoids having to specify an approximating law of motion for the transition of \( m_t \).

In particular, Den Haan (1996) parameterizes the conditional expectation. With this approximation the first-order conditions of the agent can be written as

\[
\frac{1}{c_{i,t}} = \exp \{ P_n(e_{i,t}, k_{i,t}, z_t, m_t; \lambda_p) \}, \quad \text{and} \quad (9)
\]

\[
c_{i,t} + k_{i,t+1} = r_t^{w} k_{i,t} + (1 - \tau_t) r_t^{w} e_{i,t} + \mu_t^{w} (1 - e_{i,t}) + (1 - \delta) k_{i,t}, \quad (10)
\]

where \( P_n(\cdot; \lambda_p) \) is a flexible functional form of order \( n \) with coefficients \( \lambda_p \). The algorithm works as follows.

- **Start with an initial guess for the parameterized conditional expectation, characterized by its coefficients, \( \lambda_p \).** Note that this is equivalent to having the individual policy functions for consumption and capital.

- **Simulate the economy.** Prices are calculated using the observed cross-sectional mean capital stock. Construct a time series for \( m_t \) and for one agent keep track of \([e_{i,t}, k_{i,t}, y_{i,t+1}, z_t, m_t]\), where \( y_{i,t+1} \) is defined as

\[
y_{i,t+1} = \beta \frac{r_{t+1}^{w} k_{i,t} + 1 - \delta}{c_{i,t+1}}. \quad (11)
\]

Note that

\[
\frac{1}{c_{i,t}} = \exp \{ P_n(e_{i,t}, k_{i,t}, z_t, m_t; \lambda_p) \} \approx E_t y_{i,t+1}, \quad (12)
\]

Moreover, it is no longer necessary that \( m_t \) includes the aggregate capital stock. For example, \( m_t \) could just include some percentiles.

Throughout this paper I will use \( P_n(\cdot; \lambda_p) \) to indicate a flexible functional form of order \( n \) with coefficients \( \lambda_p \). The notation will not make clear that different classes may be used for different objects. For example, the conditional expectation in 2 is approximated with a different polynomial than the cross-sectional density discussed below.

7
so one can use non-linear least squares to get a new estimate of $\lambda_p$.

- Iterate until the values for $\lambda_p$ have converged.

This procedure is similar to the one used by Krusell and Smith (1998), but does not require a specification of the transition law of motion for the moments. Of course, if one projects $y_{i,t+1}$ on $\exp \left\{ P_n(e_{i,t}, k_{i,t}, z_t, m_t; \cdot) \right\}$ then this mapping is implicitly taken into account, since $y_{i,t+1}$ depends through $r_{t+1}^k$ on next period’s cross-sectional distribution.

Solution techniques that are based on simulations are easy to program, but as will become clear in Section , also have some disadvantages.

### 3.1.3 Obtain aggregate and individual policy functions with projection procedures

Algan, Allais, and Den Haan (2006) develop an algorithm that is almost completely based on projection techniques. They also approximate the cross-sectional distribution with a finite set of moments, $m_t$, and for the individual problem, the state variables are, thus, again $e_{i,t}$, $k_{i,t}$, $z_t$, and $m_t$. A projection procedure would consist of (i) constructing a grid in the state variables, (ii) use a quadrature procedure to calculate the conditional expectation in 2, and (iii) find the coefficients of the approximating function by setting the errors on the grid equal to zero.\(^5\)

For the type of problem considered in this paper, some additional information is needed before projection procedures can be implemented. To understand why, consider a particular grid point, that is, a particular combination of $e_{i,t}$, $k_{i,t}$, $z_t$, and $m_t$. Calculation of

$$E_t \left[ \frac{\beta \gamma_{t+1}^k + 1 - \delta}{c_{i,t+1}} \right]$$

requires knowing the aggregate capital stock, $K_{t+1}$. To calculate $K_{t+1}$ at a particular grid point requires not only knowing $m_t$ (and $z_t$), but requires knowing the actual distribution. Algan, Allais, and Den Haan (2006) deal with this problem by parameterizing the cross-sectional distribution. Conditional on a particular functional form, say the exponential of an $n^{th}$-order polynomial, then the $n$ values of $m_t$ pin down a particular density. For

\(^5\)Or in case there are more grid points than coefficients by minimizing some loss criterion.
example, if one uses a second-order exponential, i.e., a normal, then the mean and the variance pin down the density. Algan, Allais, and Den Haan (2006) use a very particular approximating functional form and this will be discussed in Section 4. For now simply keep in mind that there is a mapping between the values of the cross-sectional moments, \( m \), and the coefficients of the parameterized density, \( \lambda_m \). Given the parameterization \( P_n(e, k; \lambda_m) \), calculation of the conditional expectation is straightforward and standard projection methods can be implemented.\(^6\)

The description so far assumes that the order of the approximation of the cross-sectional density is directly related to the moments included. That is, if \( n \) moments are used as state variables, then an \( n^{th} \)-order approximation is used to approximate the cross-sectional density (and vice versa). But this may be inefficient. For example, it may be the case that only first and second-order moments are needed as state variables, but that (for the particular class of approximating polynomials chosen) a much higher order approximation is needed to get the shape of the cross-sectional distribution right.

Algan, Allais, and Den Haan (2006) deal with problem as follows. Let \( m_t = [m_{1,t}, m_{2,t}] \), where \( m_{1,t} \) consists of (lower-order) moments that serve as state variables and are used to construct the grid and where \( m_{2,t} \) consists of "reference" moments. On the grid, the values of the reference moments, \( m_{2,t} \), are calculated as a function of \( z_t \) and \( m_{1,t} \) using an approximating function \( P_n(z, m_{1,t}; \lambda_r) \).\(^7\) Algan, Allais, and Den Haan (2006) find this mapping by simulating a time series for \( m_t \) but this is the only role of simulations in their algorithm. A numerical solution has to be such that the relationship between reference moments and other state variables, i.e., \( P_n(z, m_{1,t}; \lambda_r) \), that is used in solving for the individual policy rules, is consistent with the one that comes out of the simulation.

\(^6\) The procedure used in Algan, Allais, and den Haan (2008) is actually more cumbersome than necessary. They solve the individual policy rule taking as given an aggregate law of motion for the transition of the moments, \( \Gamma(z', z, m) \). Then they use the procedure described in the text to update \( \Gamma(z', z, m) \). Next, they iterate between the two problems until there is convergence in the aggregate law of motion, similar to the procedure used by Krusell and Smith (1998). But note that one does not need to specify an aggregate law of motion as an intermediate step.

\(^7\) Dependence on \( m_{1,t} \) turns out to be not important and Algan, Allais, and Den Haan (2008) simply assume that \( m_{2,t} \) takes on two different values depending on the two different values of \( z_t \).
3.2 Perturbation approaches

In this section, I discuss two perturbation procedures. The procedure developed by Preston and Roca (2006) is a "pure" implementation of the perturbation procedure. We will see that the order of the implementation used implies which moments of the cross-sectional distribution should be included. For the pure perturbation procedure of Preston and Roca (2006), the non-stochastic steady state, around which the solution is perturbed, corresponds to the model solution when both aggregate and idiosyncratic uncertainty are equal to zero. The stochastic economy always has some cross-sectional dispersion, however, and typically is at each point in time quite far from this non-stochastic steady state. Reiter (2006) avoids this criticism by developing a procedure that combines elements of projection and perturbation techniques.

3.2.1 Perturbation around scalar steady state values

Preston and Roca (2006) show how to solve models with aggregate uncertainty and heterogeneous agents with a pure perturbation procedure. The steady state they consider is the solution of the model when there is no aggregate uncertainty and no idiosyncratic uncertainty.

There are some particular—and not very important or even desirable—features of the model described above that make it less suited for perturbation procedures. So we will modify the problem slightly. The idea of perturbation procedures is to take a local approximation around the point where there is no uncertainty and then introduce the amount of uncertainty as an explicit variable in the policy function of the agent. Since perturbation techniques rely on the implicit function theorem, uncertainty should affect the problem in a smooth way. In the problem above, one can characterize the amount of uncertainty with the probability of becoming unemployed. But even an increase in the probability of becoming unemployed from zero to a slightly positive number introduces sudden discontinuous jumps in the budget set. If one wants to use a perturbation technique, it is safer to let the support of $e_{i,t}$ increase continuously with the perturbation parameter that
controls uncertainty.\textsuperscript{8} Preston and Roca (2006) assume that the law of motion for $e_{i,t}$ is given by

$$e_{i,t+1} = (1 - \rho_e)\mu_e + \rho_e e_{i,t} + \varepsilon_{i,t+1}^e$$  \hspace{1cm} (13)$$

where $\varepsilon_{i,t+1}^e$ has variance $\sigma_e^2$.\textsuperscript{9,10}

Perturbation techniques cannot deal with the type of inequality constraints discussed above because they would not show up in derivatives taken at the steady state. They can be used if the inequality constraint is replaced by a smooth penalty function that makes it costly to have low capital levels. In particular, Preston and Roca (2006) assume that there is a utility cost of holding $k_{i,t}$ in period $t$ equal to $\phi/k_{i,t}^2$.$^11$ The first-order conditions of the agents can then be written as follows

$$u_c(c_{i,t}) = \beta E_t \left[ -2\phi k_{i,t+1}^{-3} + u_c(c_{i,t+1}) (r_{t+1} + 1 - \delta) \right]$$ \hspace{1cm} (14)$$

$$k_{i,t+1} = (1 - \delta)k_{i,t} + r_t k_{i,t} + w_t e_{i,t}l - c_{i,t}.$$ \hspace{1cm} (15)$$

The order of the perturbation approximation and the set of choice variables are related to each other. If a second-order approximation is used then—as will be explained below—the state variables for the agent are $s_{i,t}$ with

$$s_{i,t} = \{k_{i,t}, e_{i,t}, a_t\}$$ \hspace{1cm} (16)$$

\textsuperscript{8}In the model discussed above this can be accomplished by letting both the probability and the drop in income increase continuously with the parameter that controls the amount of uncertainty.

\textsuperscript{9}Below, it will become clear that the analysis relies on implementing the perturbation procedure without the standard log transformation of the variables. By assuming that the laws of motion for the stochastic variables, $e_{i,t}$ and $z_t$, are linear in levels instead of logs one avoids having to take an approximation of this exogenous law of motion. As long as the uncertainty is not too large one would not run into problematic negative values, so using levels instead of log-levels wouldn’t be that bad.

\textsuperscript{10}One could allow this law of motion to depend on the aggregate state. This specification implies that aggregate labor, $L_t$, is constant.

\textsuperscript{11}Instead of assuming a utility costs one can also assume that the cost is measured in commodities. The penalty term in the Euler equation is then multiplied by $u_c(c_{i,t+1})$, which makes it less powerful because $c_{i,t+1}$ tends to be low when the agent lowers his capital holdings and it isn’t clear what will happen with the cross product.
where

\[ a_t = \{ z_t, K_t, \Phi_t, \Psi_t \}, \]  
\[ K_t = \int_0^1 (k_{i,t} - K_t) \, di, \]  
\[ \Phi_t = \int_0^1 (k_{i,t} - K_t)^2 \, di, \]  
\[ \Psi_t = \int_0^1 (k_{i,t} - K_t) (e_{i,t} - \mu_e) \, di. \]  

That is, first and second-order moments of the cross-sectional distribution are included. If a first-order approximation is used, then only first moments are included. I will return to this issue below.

Let \( h^v \) be the policy function for variable \( v \) with \( v \in \{ c, k, K, \Phi, \Psi \} \). To get the perturbation solution, we write the model as follows.

\[
\frac{1}{h^c(s_{i,t}, \sigma)} = \beta E_t \left[ -2\phi h^k(s_{i,t}, \sigma)^{-3} + \frac{(r_{t+1} + 1 - \delta)}{h^c(s_{i,t+1})} \right],
\]

\[
h^k(s_{i,t}, \sigma) = (1 - \delta) k_{i,t} + r_t k_{i,t} + w_t e_{i,t} \bar{l} - \phi k_{i,t}^{-2} - h^c(s_{i,t}, \sigma),
\]

\[
K_{t+1} = h^K(a_t, \sigma) = \int_0^1 h^k(s_{i,t}, \sigma) \, di,
\]

\[
\Phi_{t+1} = h^\Phi(a_t, \sigma) = \int_0^1 \left( h^k(s_{i,t}, \sigma) - \bar{k} \right)^2 \, di, \quad \text{and}
\]

\[
\Psi_{t+1} = h^\Psi(a_t, \sigma) = \int_0^1 \left( h^k(s_{i,t}, \sigma) - \bar{k} \right) (e_{i,t} - \mu_e) \, di.
\]

Here, \( \sigma \) is a scalar parameter that scales both types of uncertainty, \( \sigma_e \) and \( \sigma_z \). The variables \( r_t, w_t, \) and \( s_{i,t+1} \) are given by

\[
r_t = \alpha z_t (K_t / L)^{\alpha-1},
\]

\[
w_t = (1 - \alpha) z_t (K_t / L)^{\alpha}, \quad \text{and}
\]

\[
s_{i,t+1} = \left\{ h^k(s_{i,t}), \rho_e e_{i,t} + \varepsilon_{i,t+1} + \rho z_t + \varepsilon_{z,t+1}, h^K(a_t, \sigma), h^\Phi(a_t, \sigma), h^\Psi(a_t, \sigma) \right\}.
\]

Because of the aggregation constraints, it is important that the solution is specified in the untransformed level of the variables and not for example in logs.\(^\text{12}\) If not, then the

\(^{12}\text{This would not be true if one would approximate the aggregate constraints as well. But at the non-stochastic steady state, agents are equally rich and an approximation using equal weights could very well be inaccurate, given that at each point in time there typically are large differences in individual wealth levels in this type of model.}\)
functional forms of $h^K$, $h^\Phi$, and $h^\Psi$ would not be consistent with the functional forms of $h^c$ and $h^k$. The aggregation constraint makes clear what the list of state variables should be for the particular approximation order chosen. That is, a particular approximation implies a particular law of motion for the cross-sectional income and wealth distribution, which in turn implies what the relevant state variables are.

Suppose that one uses a first-order approximation. Clearly, $k_{i,t}$ and $e_{i,t}$ matter for the individual policy functions. The agent also cares about prices and, thus, about $z_t$, $K_t$, and future values of $K_t$. When the savings function is linear in $k_{i,t}$, $z_t$, and $K_t$, then the aggregation restriction (together with the linearity of the policy function) implies that $K_{t+1}$ is linear in these variables as well, and that other moments of the cross-sectional distribution, thus, should not be included.

If the individual policy functions are second-order and in particular include $(k_{i,t} - \bar{k})^2$ and $(k_{i,t} - \bar{k})(e_{i,t} - \mu_e)$, then the aggregation constraint implies that next period’s capital stock depends on $\Phi_t$ and $\Psi_{t+1}$, which means that these should be included as state variables as well.$^{13}$

Perturbation approximations specify complete polynomials.$^{14}$ This means that $(k_{i,t} - \bar{k})^2$ is not combined with any other state variables in a second-order approximation because it is a second-order term. Similary, $\Phi_t$ and $\Psi_t$ only appear by themselves since they are also second-order terms.

To see that perturbation techniques can be used, first use Equations 22 through 24 to substitute out $r_{t+1}$, $r_t$, $w_t$, and $s_{i,t+1}$. Equation 21 then specifies a set five equations in five functions $h^c(s_{i,t}, \sigma)$, $h^k(s_{i,t}, \sigma)$, $h^K(a_t, \sigma)$, $h^\Phi(a_t, \sigma)$, and $h^\Psi(a_t, \sigma)$ and one can use standard perturbation techniques to get the coefficients. The implementation described here includes both the Euler equation and the budget constraint as separate equations and results in approximations for consumption and capital of the same order. It is typically better to substitute out consumption using the budget constraint and only use the perturbation approach to get a numerical solution for capital. Consumption can then be calculated using the actual budget constraint.

$^{13}$The integral of other products can be simplified. For example, $\int (k_{i,t} - \bar{k})(K - \bar{k})di$ equals $(K_t - \bar{k})^2$.

$^{14}$A complete polynomial of order $n$ in $x$ and $y$ includes all terms $x^{n_1}y^{n_2}$ such that $n_1 + n_2 \leq n$. 

13
3.2.2 Perturbation around steady state cross-sectional distribution

The procedure of Preston and Roca (2006) perturbs around the point where there is neither aggregate nor idiosyncratic uncertainty. The idea of the procedure in Reiter (2006) is to take a perturbation around the model outcome with no aggregate uncertainty. This outcome corresponds with a (constant) cross-sectional distribution for income and capital levels and is solved with a projection method. In describing the algorithm, I develop the algorithm as a general perturbation problem and in doing so deviate from the description in Reiter (2006). But the underlying idea is the same.

Consider a numerical solution to the model of Section 2

\[ k_{i,t+1} = P_n(e_{i,t}, k_{i,t}, a_t, m_t; \lambda_k) \]  

where \( \lambda_k \) is a vector with the coefficients of the numerical solution for the consumption function. \( P_n(\cdot; \lambda_k) \) is an approximating (but fixed) functional form, say an \( n \)-th-order polynomial. Let the law of motion for \( m_t \) be given by

\[ m_{t+1} = \Gamma_{\lambda_k}(a_{t+1}, a_t, m_t). \]  

The subscript \( \lambda_k \) makes clear that this law of motion depends on the solution of the individual policy function. That is, a different individual policy rule will imply a different law of motion for the cross-sectional distribution. It is assumed that \( m_t \) is more than a limited set of moments, but pins down—possibly with additional assumptions—the complete cross-sectional distribution. For example, \( m_t \) could be the values of a histogram defined on a fine grid.\(^\text{15}\) This assumption implies that—conditional on the individual policy function—the mapping \( \Gamma_{\lambda_k} \) is known, although implementing it may require some numerical procedures like quadrature integration. In other words, given the choice to approximate the savings function with \( P_n(\cdot; \lambda_k) \) and to characterize the cross-sectional distribution in a particular way, the only unknown is \( \lambda_k \). As soon as \( \lambda_k \) is known, then all variables, including \( m_{t+1} \), can be calculated for a given set of initial values and realizations of the shock.

\(^{15}\)\( m_t \) could be a set of moments but then it has to be accompanied by a functional form assumption so that the cross-sectional density is pinned down as discussed in Section 4.3.
The individual policy function in Equation 25 can be written without the aggregate state variables, but with time-varying coefficients. That is,

\[ k_{i,t+1} = P_n(e_{i,t}, k_{i,t}; \lambda_{k,t}) \]  

(27)

with

\[ \lambda_{k,t} = \lambda_k(a_t, m_t). \]  

(28)

Let \( s = [a, m] \), let the dimension of \( \lambda_{k,t} \) be given by \( n_{\lambda_k} \), and let \( [\bar{e}, \bar{k}]' \) be an \( n_{\lambda_k} \times 1 \) vector with nodes for the employment status and capital levels.\(^{16}\) Evaluated at the nodes for the individual state variables, \( [\bar{e}, \bar{k}] \), and arbitrary values of the aggregate state variables, the first-order conditions of the agent can be written as follows.\(^{17}\)

\[
\frac{1}{(1 - \delta) + r(s)} \frac{1}{\bar{k}_j + w(s) \bar{e}_j \bar{l} - P_n(\bar{e}_j, \bar{k}_j; \lambda_k(s))} = \beta E \left[ \frac{r(s') + 1 - \delta}{((1 - \delta) + r(s')) P_n(\tilde{e}_j, \tilde{k}_j; \lambda_k(s')) + w(s) \tilde{e}' \tilde{l} - P_n(e', P_n(\tilde{e}_j, \tilde{k}_j; \lambda_k(s)); \lambda_k(s'))} \right] 
\]

(29)

In equilibrium, the endogenous part of \( s' \), i.e., \( m' \), is determined by

\[ m' = \Gamma_{\lambda_k}(a', a, m), \]  

(30)

where, as mentioned above, \( \Gamma_{\lambda_k}(\cdot) \) is—conditional on knowing \( \lambda_k(\cdot) \)—a known function.

Suppose that \( a \) is constant and that \( m' = m \) characterizes the corresponding cross-sectional distribution. Evaluated at these constant values for \( a \) and \( m \), Equation 29 is then a standard set of \( n_{\lambda_k} \) equations to solve for the \( n_{\lambda_k} \) (constant) elements of \( \lambda_k(s) \). But to understand the procedure considered here, it is important to think of Equation 29 (with \( m' \) determined by Equation 30) as a system that defines the vector-valued function \( \lambda_k(s) \).

It is important to understand what is fixed and what we are solving for in this system. First, \( P_n(\cdot; \lambda_k) \) has a known functional form, namely the one chosen as a numerical approximation. In the example considered in Section 2, the stochastic variables \( e' \) and \( a' \)

\(^{16}\)That is, I consider here the case where there are exactly enough grid points to determine the elements of \( \lambda_k \).

\(^{17}\)For simplicity we assume that unemployment benefits are zero and there is no binding constraint on capital.
have discrete support so there is an analytical expression for the conditional expectation in Equation 29. If this is not the case, then a numerical integration procedure has to be used. But for every quadrature procedure chosen, Equation 29 represents a fixed set of equations. The same is true for Equation 30. It may be possible that \( m' \) is only implicitly defined by a set of equations. This does not matter. Essential is that there is a fixed set of equations that in principle determines \( m' \).

Thus, Equation 29 (with \( m' \) determined by 30) is a system in which the coefficients of the approximating individual policy function, \( \lambda_{k,t} \), are the variables. That is, instead of consumption and capital being variables, the coefficients of the policy function have become the variables. The idea is now to solve for these functions using the perturbation approach. That is, we write \( \lambda_k(s) \) as \( h^{\lambda_k}(a,m;\Delta^a) \) and its Taylor expansion around the steady state as

\[
h^{\lambda_k}(a,m;\Delta^a) = h^{\lambda_k}(a,m;0) + h^{\lambda_k}_{a}(a,m) + h^{\lambda_k}_{m}(m) + h^{\lambda_k}_{\Delta^a} \Delta^a \\
+ h^{\lambda_k}_{aa}(a,m)^2/2 + h^{\lambda_k}_{mm}(m)^2/2 + h^{\lambda_k}_{\Delta^a\Delta^a}(\Delta^a)^2/2 \\
+ \text{second-order cross products} \\
+ \cdots
\]

As in standard perturbation procedures, we can find the coefficients of the Taylor expansion by taking successive derivatives of Equation 29.

This procedure assumes that \( m \) is more than a very limited set of moments such as the mean capital stock. The elements of \( m \) should pin down the complete cross-sectional distribution. One possibility would be to let \( m \) be the set of values of the CDF at a very fine grid. The value of \( n_{\lambda_k} \) is then very large and one has to find the policy function for many variables.\(^{18}\) This could be especially problematic if higher-order perturbation solutions are considered or if \( P_n(\cdot) \) is a non-linear function of \( m \). In this case it may be better to impose some structure on the functional form of the cross-sectional distribution so that the cross-sectional distribution is fully determined by a smaller set of coefficients. In particular, in Section Algan, Allais, and Den Haan (2006) we show how a sixth-order polynomial (whose coefficients are pinned down by six moments) describes the cross-sectional distributions

\(^{18}\)This could easily be 1,000 or more.
generated by this model through time well.

4 Simulation

In this section, I discuss different procedures to simulate an economy with heterogeneous agents using numerical solutions for the individual policy rules. The most common procedure is to use a finite number of agents and to use random number generators for both the aggregate and the idiosyncratic shocks. With a finite number of agents there will be cross-sectional sampling variation in the simulation while there is none according to the true model. Even when a large total number of agents is used, then some subgroups may still have a low number of agents and their characteristics measured with substantial sampling noise.

This section discusses three procedures to generate cross-sectional distributions of a continuum of agents that do not involve cross-sectional variation. The first two are grid methods, one of which requires the inverse of the policy function while the other does not. The third procedure uses polynomials. It imposes more structure on the functional form but uses many less coefficients to characterize the distribution and at least in the examples considered here does so quite accurately.

4.1 Grid method I: calculation of inverse required

Consider a fine grid for the capital stock. This simulation procedure approximates at each point in time the CDF with a linear spline. This means that in between grid points the distribution is assumed be uniform. Point mass at the borrowing constraint means that the value of the CDF at the first node is strictly positive. To calculate the function values at the nodes in the subsequent period, the procedure calculates the capital stocks that would have led to the value at the node or a smaller one and uses the probability of these capital stocks as the CDF value at this node.\(^{19}\) Backing out the value of capital that leads to a particular node value requires using the inverse of the policy function. Note that figuring out what the mass is of the capital values that lead to a capital choice that is

\(^{19}\)This procedure is proposed by Rios-Rull (1997), and used in Heathcote (1997), and Reiter (2002).
smaller than or equal to a particular value would be very tedious if the policy function is not monotone.

**Information used.** The beginning-of-period $t$ distribution of capital holdings is fully characterized by the following:

- the fraction of unemployed agents with a zero capital stock, $p^{u,0}_t$,
- the fraction of employed agents with a zero capital stock, $p^{e,0}_t$,
- the distribution of capital holdings of unemployed agents with positive capital holdings, and
- the distribution of capital holdings of employed agents with positive capital holdings.

The goal is to calculate the same information at the beginning of the next period. Besides these four pieces of information regarding the cross-sectional distribution one only needs (i) the realizations of the aggregate shock this period and next period and (ii) the individual policy function.

**Grid** Construct the following grid and define the beginning-of-period distribution of capital as follows.

- $\kappa_0 = 0$ and $\kappa_i = \kappa i$, $i = 1, \cdots, J$.
- Let $p^{w,0}_t$ be the fraction of agents with employment status $w$ with a zero capital stock at the beginning of period $t$.
- For $i > 0$, let $p^{w,i}_t$ be equal to the mass of agents with a capital stock bigger than $\kappa_{i-1}$ and less than or equal to $\kappa_i$. This mass is assumed to be distributed uniformly between gridpoints.

---

20Employed agents never choose a zero capital stock but some unemployed agents that chose a zero capital stock last period have become employed this period.
We have

\[
\sum_{i=0}^{J} p_t^{w,i} = 1, \quad \sum_{i=0}^{J} p_t^{e,i} = 1.
\]

Denote this beginning-of-period distribution function by \( P_t^w(k) \).

**End-of-period distribution** The first step is to calculate the end-of-period distribution of capital.

For the unemployed calculate the level of capital holdings at which the agent chooses \( \kappa_i \). If we denote this capital level by \( x_t^{u,i} \) then it is defined by\(^2\)

\[
k'(x_t^{u,i}, \cdot) = \kappa_i.
\]

Since this involves inverting the policy function, this is the hard part of the procedure. Now compute the end-of-period distribution function at the grid points as

\[
F_t^{u,i} = \int_{0}^{x_t^{u,i}} dP_t^u(k) = \sum_{i=0}^{\bar{r}_u} p_t^{u,i} + \frac{x_t^{u,i} - \kappa_{1+u}}{\kappa_{1+u} - \kappa_{1+u}} p_t^{u,\bar{r}_u+1},
\]

where \( \bar{r}_u = \bar{i}(x_t^{u,i}) \) is the largest value of \( i \) such that \( \kappa_i \leq x_t^{u,i} \). The second equality follows from the assumption that \( P_t^u \) is distributed uniformly between gridpoints.

A similar procedure is used to calculate the end-of-period distribution for the employed.

\[
F_t^{e,i} = \int_{0}^{x_t^{e,i}} dP_t^e(k) = \sum_{i=0}^{\bar{r}_e} p_t^{e,i} + \frac{x_t^{e,i} - \kappa_{1+e}}{\kappa_{1+e} - \kappa_{1+e}} p_t^{e,\bar{r}_e+1},
\]

where \( \bar{r}_e = \bar{i}(x_t^{e,i}) \) is the largest value of \( i \) such that \( \kappa_i \leq x_t^{e,i} \).

**Next period’s beginning-of-period distribution** Let \( g_{u_t^u u_{t+1}^u a_t a_{t+1}} \) stand for the mass of agents with employment status \( u \) that have employment status \( u_{t+1} \), conditional on the values of \( a_t \) and \( a_{t+1} \). For each combination of values of \( a_t \) and \( a_{t+1} \) we have

\[
g_{u_t^u u_{t+1}^u a_t a_{t+1}} + g_{e_t^u u_{t+1}^e a_t a_{t+1}} + g_{u_t^e e_{t+1}^u a_t a_{t+1}} + g_{e_t^e e_{t+1}^e a_t a_{t+1}} = 1.
\]

\(^2\) This is a non-linear problem (and has to be calculated at many nodes) but it should be a well behaved problem.
We then have

\[ p^{w,i}_{t+1} = \frac{q_{u}^{w,i+1}}{q_{u}^{w,i+1} + g_{u}^{w,i+1}} F_{t}^{w,i} + \frac{g_{e}^{w,i+1}}{q_{u}^{w,i+1} + g_{e}^{w,i+1}} F_{t}^{e,i} \]  \tag{35}

and

\[ p^{w,0}_{t+1} = P^{w,0}_{t+1} \]  \tag{36}
\[ p^{w,i}_{t+1} = P^{w,i}_{t+1} - P^{w,i-1}_{t+1} \]  \tag{37}

4.2 Grid method II: no calculation of inverse required

This method also uses a grid. Now it is assumed, however, that the distribution only has mass at the grid points. In terms of the information used, the notation, and the specification of the grid, everything is identical to the first procedure. An important advantage of this procedure is that it doesn’t require using the inverse of the policy function. Moreover, it doesn’t require the policy function to be monotone.

**End-of-period distribution** The first procedure goes through the CDF of the end-of-period distribution using the specified grid of nodes. It then works backwards to determine which beginning-of-period capital values lead to this or a smaller node value. This second procedure goes instead through the CDF of the beginning-of-period distribution, also using the specified grid of nodes. It then calculates which capital stock is chosen and assigns the probability of the beginning-of-period capital stock to grid points around the end-of-period capital choice.

Let \( f^{w,i}_{t} \) be the mass at node \( i \) of the distribution of capital for agents with employment status \( w \). It is given by

\[ f^{w,i}_{t} = \sum_{j=0}^{J} p^{w,j}_{t} \omega^{i,j}_{t}, \]  \tag{38}

where

\[ \omega^{i,j}_{t} = \begin{cases} 
0 & \text{if } k'(\kappa^{w,j}_{t}, \cdot) \leq \kappa_{i-1} \\
p^{w} \frac{k'(\kappa^{w,j}_{t}, \cdot) - \kappa_{i-1}}{\kappa_{i} - \kappa_{i-1}} & \text{if } \kappa_{i-1} < k'(\kappa^{w,j}_{t}, \cdot) < \kappa_{i} \\
p^{w} & \text{if } k'(\kappa^{w,j}_{t}, \cdot) = \kappa_{i} \\
p^{w} \frac{\kappa_{i+1} - k'(\kappa^{w,j}_{t}, \cdot)}{\kappa_{i+1} - \kappa_{i}} & \text{if } \kappa_{i} < k'(\kappa^{w,j}_{t}, \cdot) < \kappa_{i+1} \\
0 & \text{if } k'(\kappa^{w,j}_{t}, \cdot) \geq \kappa_{i-1} 
\end{cases} \]  \tag{39}
Next period’s beginning-of-period distribution  The equations for the transition from the end-of-period distribution to next period’s beginning-of-period distribution are identical to those for the first procedure.

4.3 Simulation without cross-sectional sampling variation

The original model assumes that there is a continuum of agents, a property that is important in the definition of the equilibrium and the design of several algorithms. There are several ways to simulate and avoid the cross-sectional sampling variation generated when a finite number of agents is used.

Algan, Allais, and Den Haan (2006) propose an alternative solution that is simple and fast. Suppose that the cross-sectional density is given by a particular density, \( P(k; \rho_1) \), where \( \rho_1 \) are the coefficients of the density characterizing the density in period 1. To simplify the exposition I ignore the employment status.\(^{22}\) \( P(k; \rho_1) \) together with individual policy rules are in principle sufficient to determine \( P(k; \rho_2) \). Algan, Allais, and Den Haan (2006) propose the following procedure. Let \( P(k; \rho) \) be an \( n^{\text{th}} \)-order polynomial. Use \( P(k; \rho_1) \) together with individual policy rules to determine the first \( n \) period 2 moments of \( k, [m(1), \cdots m(n)] \) Standard quadrature methods can be used. Then find the values of \( \rho_2 \) to ensure that the moments of \( P(k; \rho_2) \) are consistent with these moments. Iterating generates a complete time series.

The tricky part is to find the coefficients that correspond with a set of specified moments. Algan, Allais, and Den Haan (2006) make this problem substantially easier by using a particular functional form for \( P(k; \rho) \). In particular, they use

\[
P(k; \rho_t) = \rho_{t,0} \exp \left( \begin{array}{c}
\rho_{t,1} [k - m(1)] \\
\rho_{t,2} [(k - m(1))^2 - m(2)] \\
\vdots \\
\rho_{t,n} [(k - m(1))^n - m(n)]
\end{array} \right).
\]  \(40\)

When the density is constructed in this particular way, the coefficients, except for \( \rho_0 \),
can be found with the following minimization routine:

$$\min_{\rho_t, 1: \rho_t, 2: \cdots, \rho_t, n} \int_0^\infty P(k, \rho_t) dk. \quad (41)$$

The first-order conditions correspond exactly to the condition that the first $n$ moments of $P(k, \rho_t)$ should correspond to the set of specified moments. $\rho_{t,0}$ does not appear in these equations, but it is determined by the condition that the density integrates to one.

The Hessian (times $\rho_0$) is given by

$$\int_0^\infty X (m(1), \cdots, m(n)) X (m(1), \cdots, m(n))^\prime P(k, \rho) dk, \quad (42)$$

where $X$ is an $(n \times 1)$ vector and the $i^{th}$ element is given by

$$\begin{align*}
(k - m(1)) & \quad \text{for } i = 1 \\
(k - m(1))^{i} - m(i) & \quad \text{for } i > 1
\end{align*} \quad (43)$$

The Hessian is positive semi-definite since $X$ does not depend on $\rho^{w}. 23$ Consequently, this is a convex optimization problem and, thus, avoids the need for good initial conditions. 24

An advantage of this simulation procedure is that it can easily be extended to densities of multiple variables. It may be less appropriate if the cross-sectional distribution displays discontinuous jumps. As long as marginal propensities to save are locally relatively stable, however, the smoothness of the approximating density would not matter. In fact, in this type of model marginal propensities do not vary much globally unless one gets close to the borrowing constraint. Nevertheless, this is something one should keep in mind. Especially, if results do not settle down if a higher-order approximation is used, then one should consider using a piecewise linear distribution function.

4.4 Comparison of simulation methods

- The importance of sampling variation

- Can you use smooth polynomial approximations when the CDF has jumps?

23Note that evaluated at the solution for $\rho^{w}$, it is a covariance matrix.
24When using exponentials to parameterize the density and using an equation solver to find the coefficients, the algorithm often got stuck and had to be restarted with better initial conditions.
5 Aggregation

- Exact and approximate aggregation

- Young 2005
  - Does approximate aggregation imply you can use a representative agent model? KS2006 - not for asset pricing in particular the risk free rate
  - model above: a representative agent model with exact same preferences and average employment productivity has aggregate behavior that is very similar to what is observed here. That is, precautionary savings is relatively low (also see Carroll 2000). This does not mean that there is full consumption insurance. Individual consumption is still more volatile than aggregate consumption. In terms of utility however the agent doesn’t care much.
  - The extension with stochastic discount factors still has approximate aggregate but the representative agent economy would look different (at least if the same preferences are used). Chang and Kim 2006 IER has approximate aggregation but "produces aggregate time series behavior that is different from that in either a standard representative agent model with highly inelastic labor supply or in a Hansen-Rogerson economy".

- What PR say on page 3 about the second-order properties being quantitatively more important for the mean than precautionary savings motives should probably discussed here as well.

- Krueger and Kubler 2004 and aggregation in OLG framework but compare with Kjetil’s paper.

6 Accuracy

Models with heterogeneous agents and aggregate uncertainty are complex models. As was pointed in Section 2, it is not even clear for which class of models a recursive equilibrium
exists for the set of state variables typically used in numerical analysis. This by itself
would mean that careful accuracy tests are required. Another reason is that simulations
play a central or minor role in several algorithms. Simulations are inefficient numerical
tools because sampling uncertainty disappears at a slow rate and because simulated data
tend to cluster, whereas accuracy requires data to be spread out.

Given the complexity of the model there are several aspect that needs to be checked.
Although it depends somewhat on the algorithm, typically the individual policy functions
and the aggregate policy functions need to be checked for accuracy. The best way to check
for the accuracy of individual policy functions, is to calculate for Euler equation errors
on a very fine grid and to report maximum errors expressed as a percentage of the implied
consumption choice.25

In checking for accuracy of aggregate policy functions, the numerical literature did
not provide explicit guidance. Some authors are clearly aware of the difficulty in assessing
accuracy. Krusell and Smith (1997,1998), for example, perform a variety of accuracy tests,
try out several different alternative approximating functional forms, and perform a careful
economic analysis to explain why their preferred numerical solution, one in which only
the mean matters for aggregate dynamics, is an accurate one. Unfortunately, regarding
the accuracy tests Krusell and Smith (1998) put most emphasis on the R-square, $R^2$,
and the standard error of the regression, $\sigma_u$, and the subsequent literature has treated
these as sufficient statistics to determine the accuracy of the aggregate law of motion.
Moreover, typically no results are reported for richer approximating functional forms nor
any economic motivation for why the simpler approximating functional forms would make
sense. Den Haan (1997) shows that the $R^2$ and the standard error are very weak accuracy
test and gives examples in which numerical solutions with an $R^2$ in excess of 0.9999 can still
be inaccurate. The accuracy of many results in the literature is, thus, still undocumented.

To focus the discussion, suppose that a researcher is interested in assessing the accuracy
of the following approximating law of motion

$$m_{t+1} = \bar{\alpha}_0 + \bar{\alpha}_1 m_t + \bar{\alpha}_2 a_t,$$

(44)

where \( m_t \) is the mean of the cross-sectional distribution and \( a_t \) is an aggregate shock. The researcher can also simulate this economy. That is, if \( f_t \) is the density of the cross-sectional distribution in period \( t \), then the researcher can generate the economy using the following law of motion

\[
f_{t+1} = \Upsilon(a_{t+1}, a_t, f_t).
\] (45)

Typically, there is some numerical error in this simulation, for example, because only a finite number of agents is used. To focus the discussion on what matters, these are ignored for now. Let \( m_{t+1}^f \) be the mean implied by \( f_t \). The approximating law of motion in Equation 44 would, thus, be accurate if \( m_{t+1} \) is "close to" \( m_{t+1}^f \). This is an uncontroversial statement. It is also empty because it doesn’t make clear how the comparison is done.

The standard tests of the literature, the \( R^2 \) and \( \hat{\sigma}_u \) are based on errors defined

\[
u_{t+1} \equiv m_{t+1}^f - m_{t+1} = m_{t+1}^f - (\bar{a}_0 + \bar{a}_1 m_t + \bar{a}_2 a_t).
\] (46)

That is, in calculating the prediction according to the approximating law of motion the observation generated by the true \( dgp \) is used. That is, even if the approximating law of motion would want to push the observations in a different direction, the true \( dgp \) is used each period to correct the approximating law of motion. This is the most troublesome feature of these two accuracy measures. But these measures have other problems. It is typical in the numerical solutions literature to focus on maximum errors but the \( R^2 \) and \( \hat{\sigma}_u \) are based on averages. Moreover, the \( R^2 \) inflates the measure of fit by scaling it with the variance of the dependent variable.\(^{26}\)

Den Haan (2007) proposes a much more powerful accuracy test, which is also likely to be more insightful in determining where and why the approximation fails. It consists of the following steps.

- Generate a time series for \( a_t \) and pick an initial condition \( f_1 \). This draw for \( a_t \) should not be the same draw as the one used to calculate the approximating law of motion.

\(^{26}\)If one would focus on \( m_{t+1} - m_t \) and redefine the approximating law of motion accordingly as \( m_{t+1} - m_t = \bar{a}_0 + (\bar{a}_1 - 1) m_t + \bar{a}_2 a_t \), then of course nothing has changed. But the \( R^2 \) would typically be substantially lower.
Generate a time series using Equation 45. In a typical application this would involve to simulate a panel given a particular numerical solution for the individual policy rule.

Generate a time series using the approximating law of motion given in 44. Although this series is based on the same draw for \( a_t \) and the same initial condition \( m^f_1 \), this series is not influenced by \( Y(\cdot) \).

Define the error term as \( m^f_{t+1} - m_{t+1} = m^f_{t+1} - (\hat{\alpha}_0 + \hat{\alpha}_1 m_t + \hat{\alpha}_2 a_t) \). That is, for each period \( t \) the lagged values used to generate \( m_{t+1} \) are generated by the approximating law of motion itself.

Report the maximum error. If \( m_t \) is something like the log of capital no scaling is necessary. Otherwise the author should think about appropriate scaling.

Plot the two generated series. Check in particular whether one series is systematically below the other and determine in which part of the state space the deviations are biggest. This is referred to in Den Haan (2007) as the "essential accuracy plot".

Den Haan (2007) shows that this accuracy test is just as powerful as one of the tests considered by Krusell and Smith (1997, 1998), namely the maximum 100-quarter ahead forecast error. The advantage of the "essential accuracy plot" is that it provides some useful insights. For example, suppose the economy is simulated with a finite number of agents, which means that \( m^f_{t+1} \) is actually generated with error. If one would find that \( m^f_{t+1} \) fluctuates around \( m_{t+1} \) then the deviations are likely to be due to errors in generating \( m^f_{t+1} \) not in the approximating law of motion. The essential accuracy plot would quickly make this clear.

A careful accuracy analysis consists of more than simply reporting the results of some accuracy tests. It would involve a robustness analysis to see whether the properties of

\[27\text{It is obviously cleaner to use a fresh draw. Typically it is not very important, however, since long samples are used to estimate the coefficients of the approximating law of motion.}\]
the model change if alternative specifications for the approximating law of motion are used. It is also important to "play around" to understand the differences between the approximating law of motion and the dgp it is supposed to approximate. For example, in addition to comparing $m_{t+1}$ and $m^f_{t+1}$ for a typical draw of $a_t$ one can compare the generated series for an extreme draw. It also may be good to compare impulse response functions, that is, consider the draw where $a_t$ is shocked out of its steady state value but then returns to it.

Given the complexity of these types of models and because of the interaction between aggregate transition laws and individual policy rules, it is hard to understand the role of heterogeneity. Accuracy tests may (or may not) be reassuring, but often provide limited insights into why a particular approximation works or doesn't work. Krusell and Smith (2006) stress the importance of understanding the relationship between individual policy rules and aggregate laws of motion. They propose to study two environments that are much easier to solve and are likely to give important insights. The first is the economy without aggregate uncertainty. The second is a two-period version of the model in which one can vary the cross-sectional distribution exogenously.

7 Discussion

- Accuracy is more important at this point then discussion on efficiency. Of course, it is very troublesome that a careful assessment of accuracy is not taking place (compare for example the concern in KS 1998 and more recent articles.

- Nevertheless some discussion on likely merits of different algorithm seem in place.

- Monte Carlo integration

- Which nodes

- Rectangular grid

- Inequality constraints
Perturbation - around which point to approximate. Same problem in New-Keynesian models.

Preston & Roca because its solution is more analytical may be better suited for (i) optimal policy analysis, (ii) for understanding the role of heterogeneity, and (iii) estimation. For example the discussion on page 22 makes clear the differences between the representative agent model and the heterogeneous agent model. The constant higher in the heterogeneous agent model (more precautionary savings) but the first-order coefficients are identical.

Many stochastic state variables is still a problem. Not or less so for Preston & Roca. Probably also for Reiter

8 Concluding comments

References


