Filtering Data using Frequency Domain Filters

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Fourier Transform

Given a sequence $\{x_j\}_{-\infty}^{\infty}$ the Fourier transform is defined as

$$F(\omega) = \sum_{j=-\infty}^{\infty} x_j e^{-i\omega j}$$

If $x_j = x_{-j}$ then

$$F(\omega) = x_0 + \sum_{j=1}^{\infty} x_j \left(e^{-i\omega j} + e^{i\omega j}\right) = x_0 + \sum_{j=1}^{\infty} 2x_j \cos(\omega j)$$

and the Fourier transform is a real-valued symmetric function.

Inverse Fourier Transform

GIven a Fourier Transform $F(\omega)$, one can back out the original sequence using

$$x_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)e^{i\omega j}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)\left(\cos\omega j + i\sin\omega j\right)d\omega$$

and if $F(\omega)$ is symmetric then

$$x_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)\cos\omega j\ d\omega = \frac{1}{\pi}\int_{0}^{\pi} F(\omega)\cos\omega j\ d\omega$$

Thinking differently about a time series

Fourier transform of $\{x_t\}_{t=1}^{T}$, scaled by $\sqrt{T}$

$$\tilde{x}(\omega) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} e^{-i\omega t} x_t.$$

Let

$$\omega_j = (j-1)2\pi/T,$$

The *finite* inverse Fourier transform is given by

$$x_t = \frac{1}{\sqrt{T}} \sum_{\omega_j} e^{i\omega_j t} \tilde{x}(\omega_j).$$

Thinking differently about a time series

Using

$$\tilde{x}(\omega) = |\tilde{x}(\omega)| \, e^{i\phi(\omega)}.$$

gives

$$x_t = \frac{1}{\sqrt{T}} \left(\tilde{x}(0) + 2 \sum_{\omega_j < \pi} |\tilde{x}(\omega_j)| \cos(\omega_j t + \phi(\omega_j)) \right)$$

Thinking differently about a time series

Variance of x_t for different frequencies related to

$$(|\tilde{x}(\omega)|)^2 \text{ or } \tilde{x}^2(\omega)$$

This is basically the spectrum

Spectrum

GIven a sequence $\{\gamma_j\}_{-\infty}^{\infty}$ of autocovariances of a scalar process then the Spectrum is defined as

$$S(\omega) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma_j e^{-i\omega j} = \frac{1}{2\pi} \left(\gamma_0 + \sum_{j=1}^{\infty} 2\gamma_j \cos(\omega j) \right)$$

And according to the inverse

$$\gamma_0 = \int_{-\pi}^{\pi} S(\omega) \; d\omega$$

Spectrum of filtered series

$$y_t = \sum_{j=-\infty}^{\infty} b_j x_{t-j} = b(L)x_t$$

Then

$$S_y(\omega) = b(e^{-i\omega})b(e^{i\omega})S_x(\omega) = \left|b(e^{-i\omega})\right|^2 S_x(\omega)$$

- $|\cdot|$ is the modulus of the complex number
- Note that $b(e^{-i\omega})$ is the Fourier transform of the b_j sequence
- For symmetric filters we have $b(e^{-i\omega}) = b(e^{i\omega})$

Band-pass filters

$$y_t = b(L)x_t$$

Goal:

$$S_y(\omega) = \begin{cases} S_x(\omega) & \text{if } \omega_1 \leq \omega \leq \omega_2 \\ 0 & \text{o.w.} \end{cases}$$

Thus we need

$$b(e^{-i\omega}) = \begin{cases} 1 & \text{if } \omega_1 \leq \omega \leq \omega_2 \\ 0 & \text{o.w.} \end{cases}$$

- How to find the coefficients b_j that correspond with this?
- Since $b(e^{-i\omega})$ is a Fourier transform we can use the inverse of the Fourier transform

Coefficients of band-pass filters

Inverse of the Fourier transform:

$$\begin{aligned}
b_j &= \frac{1}{2\pi}\int_{-\pi}^{\pi} b(e^{-i\omega})e^{i\omega j}d\omega \\
&= \frac{1}{2\pi}\left(\int_{-\omega_2}^{-\omega_1} 1\times e^{i\omega j}d\omega + \int_{\omega_1}^{\omega_2} 1\times e^{i\omega j}d\omega\right) \\
&= \frac{1}{2\pi}\left(\int_{\omega_1}^{\omega_2} \left(e^{i\omega j} + e^{-i\omega j}\right) d\omega\right) \\
&= \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} 2\cos(\omega j) d\omega \\
&= \frac{1}{\pi}\frac{1}{j}\sin \omega j]_{\omega_1}^{\omega_2} = \frac{\sin(\omega_2 j) - \sin(\omega_1 j)}{\pi j}
\end{aligned}$$

Using l'Hopital's rule for $j = 0$ we get

$$b_0 = \frac{\omega_2 - \omega_1}{\pi}$$

An aside on filters that induce stationarity

- If x_t is I(1) then
$$(1 - L)x_t = z_t$$
with z_t an I(0) process.
- Filtering gives
$$x_t^f = b(L)x_t$$
- Question: When is x_t^f I(0)?

An aside on filters that induce stationarity

Suppose that

$$b(L) = (1 - L)\bar{b}(L)$$

and

$$\bar{b}(1) < \infty$$

Then $x_t^f = b(L)x_t$ is stationary even if x_t is I(1)

$$\begin{aligned} x_t^f &= b(L)x_t \\ &= (1 - L)\bar{b}(L)x_t \\ &= (1 - L)\bar{b}(L)\frac{z_t}{(1 - L)} \\ &= \bar{b}(L)z_t \end{aligned}$$

Properties of the band-pass filter

$$b(L) = \sum_{j=-\infty}^{\infty} b_j L^j$$

- $b(L)$ is a polynomial of L. Consider the roots to the problem:

$$b(L) = 0$$

If $L = 1$ is a root of the problem, then we have

$$b(L) = (1 - L)\bar{b}(L) \quad \text{with } \bar{b}(1) < \infty$$

Properties of the band-pass filter

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  - But $L = 1$ is a root of our filter as long as $\omega_1 > 0$, because then we have by construction
  $$b(1) = b(e^{-i0}) = 0$$
  Clearly, if you do not filter out the zero frequency then you do not induce stationarity

More on I(1) processes

- Discussion above showed

$$x_t^f = b(L)x_t \text{ is stationary even if } x_t \text{ is I}(1)$$

- This is not enough to show that the filter does what it is supposed to do, which is
  - ensure the spectrum of the filtered series is zero for the excluded frequencies
  - ensure the spectrum of the filtered series equals the spectrum of the original series for the included frequencies
- The second condition requires a definition of the spectrum for I(1) processes

Spectrum for I(1) processes

Consider an arbitrary I(1) process

$$x_t = \frac{z_t}{1-L}$$

Let

$$x_{\rho,t} = \frac{z_t}{1-\rho L}$$

For $\rho < 1$ the spectrum of $x_{\rho,t}$ is well defined

$$S_{\rho,x}(\omega) = \frac{1}{1-2\rho\cos(\omega)+\rho^2} S_z(\omega)$$

Define the spectrum of x_t as

$$S_x(\omega) = \lim_{\rho \longrightarrow 1} S_{\rho,x}(\omega)$$

This is well defined for all $\omega > 0$, but not for $\omega = 0$.

Filtered I(1) process

$$x_t^f = b(L)x_t$$

Let $b(L)$ be a band-pass filter, that is,

$$b(e^{-i\omega}) = \begin{cases} 1 & \text{if } \omega_1 \leq \omega \leq \omega_2 \\ 0 & \text{o.w.} \end{cases}$$

Filtered I(1) process

- if $\omega_1 > 0$, then it can be shown that
  - x_t^f is stationary (because as shown above we know that $b(1) = 0$) and
  - $S_{x^f}(\omega) = \begin{cases} S_x(\omega) & \text{if } \omega_1 \leq \omega \leq \omega_2 \\ 0 & \text{o.w.} \end{cases}$
- That is, using the definition of the Spectrum for I(1) processes the filter does exactly what it is supposed to do
- Proof is simple; The only tricky thing is to prove is that

$$b(e^{-i0})S_x(0) = 0$$

Practical Filter

- The filter constructed so far is two-sided and infinite order
- Implementable version would be to use

$$\tilde{b}(L) = \sum_{j=-J}^{J} b_j L^j$$

But it is not necessarily the case that

$$\tilde{b}(1) = 0$$

So instead use

$$a(L) = \sum_{j=-J}^{J} a_j L^j$$

with

$$a_j = b_j + \mu \quad \text{and} \quad \mu = -\frac{\sum_{j=-J}^{J} b_j}{2J = 1}$$

Hodrick-Prescott Filter

- With $\lambda = 1,600$ the HP filter is approximately equal to a band-pass filter with $\omega_1 = \pi/16$ and $\omega_2 = \pi$. That is, it keeps that part of the series associated with cycles that have a period less than 32 ($=2\pi/(\pi/16)$) periods (i.e. quarters).