

## Equation Chapter 1 Section 1 The Regression Discontinuity Design

Sometimes whether something happens to you or not depends on your ‘score’ on a particular variable. You get a scholarship if you get above a certain mark in an exam, you get given remedial education if you get below a certain level, a policy is implemented if it gets more than 50% of the vote in a ballot, your sentence for a criminal offence is higher if you are above a certain age (an ‘adult’) etc etc.

All of these examples are candidates for an application of the regression discontinuity design. The essential element of a regression discontinuity design is that the probability of assignment to treatment depends in a discontinuous way on some *observable* variable  $W$ .

The simplest (and most common) form of the RDD has assignment to treatment being based on  $W$  being above some critical value  $w_0$  - I will use this case in what follows.

Note that, in some sense, the method of assignment to treatment is the very opposite here to that in random assignment – it is a deterministic function of some observable variable. But, it turns out that, in a sense I will explain, assignment to treatment is as ‘good as random’ in the neighbourhood of  $w_0$ , the discontinuity.

The basic RDD estimator can be understood very simply. Suppose we consider individuals with  $W$  in the interval  $[w_0 - \delta, w_0)$ . These are all in the control group so the outcome we will observe for these is  $y_0$ . Suppose that the expected value of  $y_0$  given  $W$  can be written as:

$$E(y|W, X = 0) = g_0(W) \quad (1)$$

Take a first-order Taylor series expansion of this about the point  $W = w_0$ . We can then write:

$$g_0(w_0 - \delta) \approx g_0(w_0) - \delta g_0'(w_0) \quad (2)$$

Hence for the group of people with  $W$  in the interval  $[w_0 - \delta, w_0)$  we will have approximately that:

$$E(y|w_0 - \delta \leq W < w_0) \approx g_0(w_0) - g_0'(w_0) E(\delta|w_0 - \delta \leq W < w_0) \quad (3)$$

Now do the same exercise for individuals with  $W$  in the interval  $[w_0, w_0 + \delta]$ . These are all in the treatment group so the outcome we will observe for these is  $y_1$ . Suppose that the expected value of  $y_1$  given  $W$  can be written as:

$$E(y|W, X = 1) = g_1(W) \quad (4)$$

Take a first-order Taylor series expansion of this about the point  $W = w_0$ . We can then write:

$$g_1(w_0 + \delta) \approx g_1(w_0) + \delta g_1'(w_0) \quad (5)$$

Hence for the group of people with  $W$  in the interval  $[w_0, w_0 + \delta]$  we will have approximately that:

$$E(y|w_0 \leq W \leq w_0 + \delta) \approx g_1(w_0) + g_1'(w_0) E(\delta|w_0 \leq W \leq w_0 + \delta) \quad (6)$$

Now consider the difference in means for the outcome for individuals the two sides of the discontinuity: by taking the difference between (3) and (6) we will have that

$$E(y|w_0 \leq W \leq w_0 + \delta) - E(y|w_0 - \delta \leq W < w_0) \\ \approx g_1(w_0) - g_0(w_0) + g_1'(w_0)E(\delta|w_0 \leq W \leq w_0 + \delta) - g_0'(w_0)E(\delta|w_0 - \delta \leq W < w_0) \quad (7)$$

Now, take the limit as  $\delta \rightarrow 0$ . The expectations on the right-hand side of (7) will both go to zero and we will be left with:

$$E(y|w_0 \leq W \leq w_0 + \delta) - E(y|w_0 - \delta \leq W < w_0) \rightarrow g_1(w_0) - g_0(w_0) \quad (8)$$

You should recognise the right-hand side as the treatment effect at  $W = w_0$ . So the RDD estimator says that you should simply compare the outcome of people who are just on both sides of the discontinuity and the difference in means between these two groups is an estimate of the treatment effect at the discontinuity. Note that we can say nothing about the treatment effect away from the discontinuity and this is a limitation of the RDD effect.

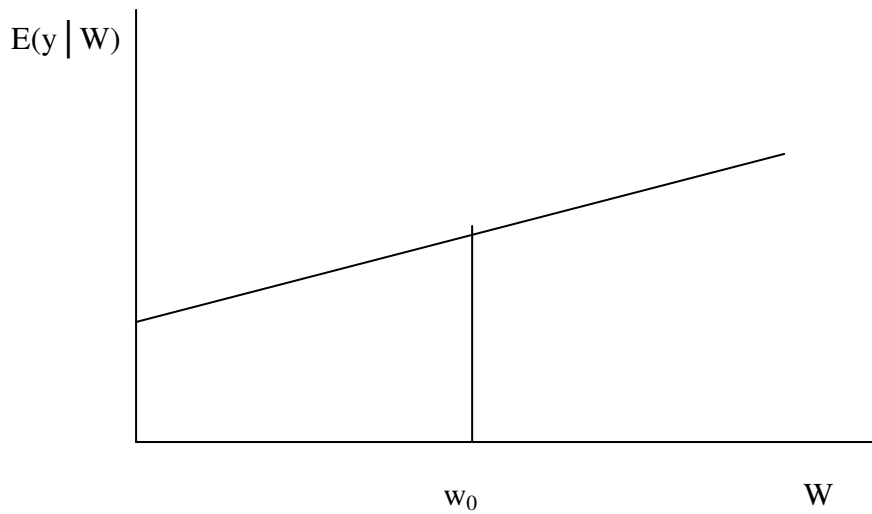
There is one important assumption behind the estimator I have described that I have hidden. The basic idea behind the RDD estimator is that people close to the discontinuity have very similar values of  $W$  but very different values of  $X$ . Any discontinuity in the outcome is ascribed to the discontinuity in  $X$ . But, this implicitly assumes that the outcome is continuous in  $W$  for both treatment and control groups. If this is not the case then the difference in outcomes at  $W = w_0$  could be the result of a discontinuity in  $g_0(W)$  and/or  $g_1(W)$ . For the mathematically inclined, when I took the limits as  $\delta \rightarrow 0$ , I was taking one-sided limits – from below for the control group and from above for the treatment group. If the outcome functions are not continuous at  $w_0$  then I will not estimate the right-hand side of (8) but

$g_1^+(w_0) - g_0^-(w_0)$  i.e. a combination of the treatment effect and the discontinuity. In most practical applications this is not a serious concern: we are happy to assume that the underlying relationship between  $y$  and  $W$  is a continuous one. Just to summarize the assumption needed for a regression discontinuity design to be useful is the following:

**Assumption RD1:**  $g_0(W)$  and  $g_1(W)$  are continuous in  $W$  at  $W = w_0$ .

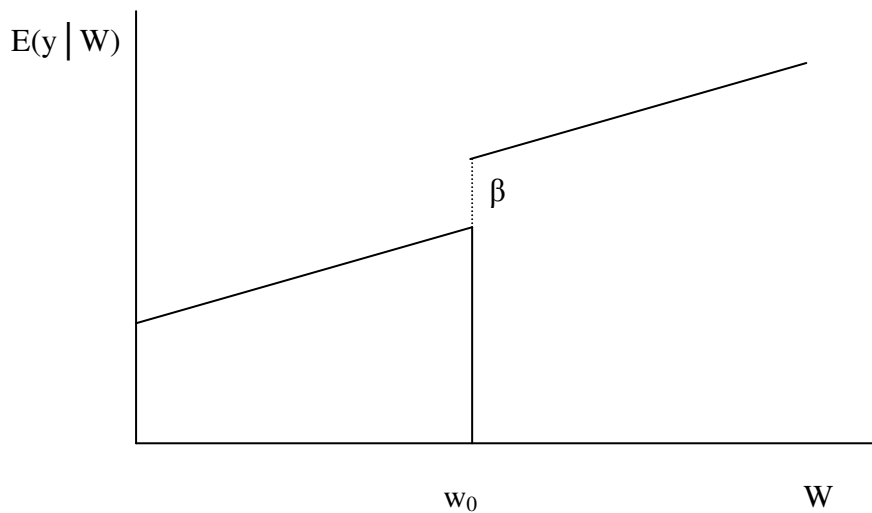
The validity of estimates based on a RDD depend critically on this assumption – so is it testable? The answer is generally ‘no’ though it is very plausible in some cases. One can think of it as an identification assumption and identification assumptions can not be fully tested using the data at hand. But if it is true then one can show that in the vicinity of  $W = w_0$  treatment status should be independent of any characteristics so is as good as randomly assigned – this is testable.

A few diagrams may make this easier to understand. To make things very easy assume that  $g_0(W)$  is linear in  $W$  and that  $g_1(W) = g_0(W) + \beta$  so that the treatment effect is identical for everyone. First, suppose that there is no treatment for anyone - then the relationship between the average value of  $y$  and  $W$  will look like:



Note that this regression line will be continuous in  $W$ .

Now introduce the treatment for all those with  $W$  above  $w_0$ . The regression line will now look like:



The regression line now has a discontinuity at  $W = w_0$  and the size of this discontinuity is the average treatment effect. Note that, at first sight, the RDD design does not seem very promising as a way to estimate a causal effect as the variable  $W$  is likely to be correlated with the outcome variable,  $y$ , and treatment can be perfectly explained by  $W$  (so there is a sort of perfect collinearity between treatment and regressor though the relationship is not a linear one).

How should one implement this in practice? Take the procedure I have described very literally and one should choose a value of  $\delta$  that is very small. This will result in a very small number of observations – you will have a consistent estimate of the treatment effect but the standard error will be very high (the precision will be low) because the number of observations is small.

So the desire to increase the sample size leads one to choose a larger value of  $\delta$ . But, there are then dangers because the last two terms in (7) can no longer be assumed to be zero. As can be seen from (7) the size of these terms depends on the derivative of the outcome functions with respect to  $W$ .

Suppose we had the situation in the Figure above but just used the naïve RDD estimator. As one increases  $\delta$  the measure of the treatment effect will get larger. This is spurious so what should one do about it. The basic idea is that one should control for the underlying outcome functions.

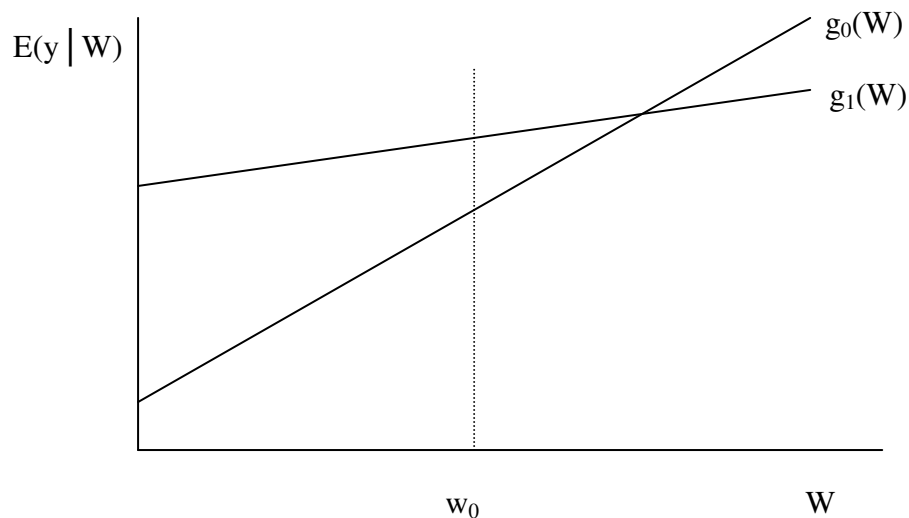
If the linear relationship is the correct specification then one could estimate the ATE simply by estimating the regression:

$$y = \gamma_0 + \beta X + \gamma_1 W + \varepsilon \quad (9)$$

Note that one can estimate this because, although  $X$  is a deterministic function of  $W$ , it is not a linear function of  $W$ .

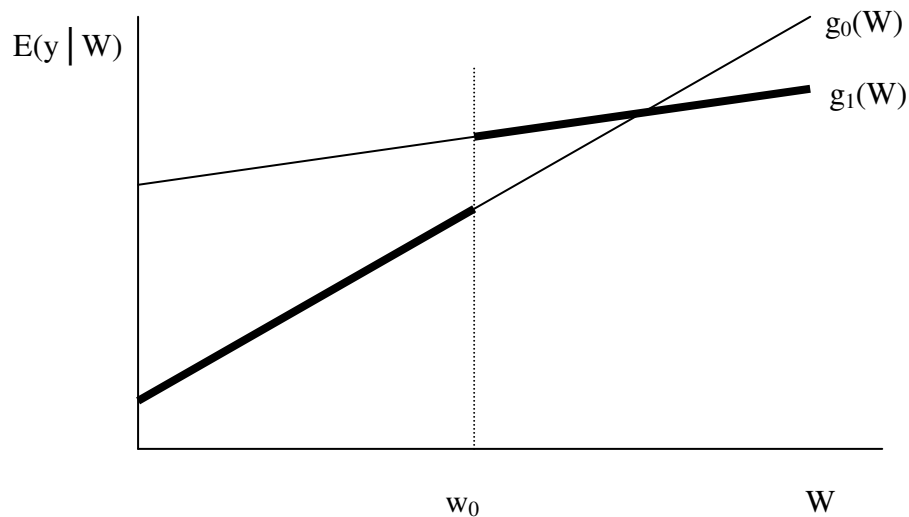
But this example has assumed that  $y$  is a linear function of  $W$ . This will typically not be the case and it is certainly true that we normally have no good reason for it to be the case.

But the relationship does not necessarily look like this. The following Figure plots a possible  $g_0(W)$  and  $g_1(W)$



The average treatment effect for a given value of  $W$  is given by the difference between the lines – as drawn this is positive for those with low values of  $W$  but actually negative for those with high values.

What will the observed value of  $y$  given  $W$  look like – this will be the bolder line in the following picture.



In this case one would want to control for a different relationship between  $y$  and  $W$  for the treatment and control groups.

Of course, the outcome functions might not be linear in  $W$  – it could be quadratic or something else. The researcher then typically faces a trade-off:

- choose a large value of  $\delta$  to get more precision from a larger sample size but run the risk of a misspecification of the underlying outcome function leads to inconsistent estimates.
- Choose a flexible underlying functional form at the cost of some precision (intuitively a flexible functional form can get closer to approximating a discontinuity in the outcomes).

In practical applications, it is usual for the researcher to summarize all the data in the graph of the outcome against  $W$  to get some idea of the appropriate functional forms and how wide a window should be chosen. But its always a good idea to investigate the sensitivity of estimates to alternative specifications.