Information Structures in Optimal Auctions*

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Abstract

A seller wishes to sell an object to one of multiple bidders. The valuations of the bidders are privately known. We consider the joint design problem in which the seller can decide the accuracy by which bidders learn their valuation *and* to whom to sell at what price. We establish that optimal information structures in an optimal auction exhibit a number of properties: (i) information structures can be represented by monotone partitions, (ii) the cardinality of each partition is finite, (iii) the partitions are asymmetric across agents. We show that an optimal information structure exists.

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1 Introduction

The optimal design of an auction has received considerable attention in the economics literature. Myerson (1981) constitutes the seminal paper in the field. Myerson shows which auction rules achieve the largest revenues to the seller in a single object auction. Most of the subsequent literature on mechanism design maintains the assumption that the information held by market participants is given as exogenous. Little is known about optimal mechanisms when the information of the participants is allowed to be endogenous.

This paper considers the optimal auction design problem when the seller can determine bidders' information precision. We consider a problem in which a seller offers a single object to a number of risk neutral bidders. The seller wishes to maximize revenues from the sale. Bidders' valuations for the object are private and not known prior to the bidding. The seller controls the bidders' information structures which generate the bidders' private information. The information structure determines the accuracy with which buyers learn their valuations prior to the auction. The seller may assign an information structure that informs a bidder perfectly or an information structure that gives the bidder only a rough guess about her true value for the object. The seller's choice of information structure is made prior to the auction and does not involve transfer payments from the bidders. After the choice of information structure by the seller, the bidders then report their value estimate to a revelation mechanism which determines the probability of winning the object and a transfer payment for every bidder. We study information structures and revelation mechanisms that maximize the seller's revenues. The solution in Myerson (1981) arises in our model as a special case when the seller informs the bidders perfectly.

We analyze the optimal information and mechanism design problem under strong informational assumptions. We assume that the seller has full control in his choice of the information structure and there is no cost to adopt a particular information structure. Our set-up allows us to emphasize two opposing effects that determine the endogenous choice of the precision of information: first, more information increases the efficiency of the auction and thus seller's revenues; second, more information increases the rents of the bidders in form of information rents which lower the seller's revenues. We analyze this trade-off and characterize the properties of optimal information structures.

The model assumes that the optimal information and mechanism design is subject to the interim incentive and interim individual rationality constraints of the bidders. By imposing the interim individual rationality constraint, each bidder is allowed to assess the value of the transaction conditional on his private information. In particular, this means that the seller cannot request payment for the private information, separately from the auction of the object itself. The adoption of the interim individual rationality constraints here can be motivated by a temporal distinction between the adoption of a set of rules governing an auction or an entire series of auction and the actual auction event. Within such a sequencing context, the individual rationality constraints arises naturally at the interim stage.[1]

Empirical applications that share features with some of our assumptions can be given, but we wish to emphasize that we are not aware of an application that fits our assumptions precisely. Our study makes strong assumptions and our results may not be directly applicable for auction design in practice. In light of the results, we shall discuss the role of the assumptions in detail in the final section.

The linkage principle of Milgrom & Weber (1982) is related to our work but obtained in a distinct informational setup. In a symmetric model with affiliated values, they show that the seller can increase revenue by releasing information publicly to all bidders. The public information reduces the winner's curse and hence the information rent of the winning bidder. In contrast, with private values, an increase in information to an individual bidder increases that bidder's information rent. While we consider the choice of information structure by the seller, a related literature considers the incentives of the buyers, to obtain more information, e.g. Cremer & Khalil (1992), Persico (2000), and Bergemann & Välimäki (2002).

Our paper is organized as follows: Section 2 describes the model. Section 3 considers the example of bidders with uniformly distributed valuations on the unit interval. Section 4 analyzes the optimal information structure when the signal space is finite. We show that: (i) the optimal information structures are partitions, (ii) the optimal partitions are asymmetric, and (iii) optimal partitions exist. Section 5 extends the characterization results to the class of all measurable information structures (possibly with infinite and uncountable

[1] We thank an anonymous referee for suggesting this point of view.

signals) and shows that the above characterization results remain to hold, in particular the optimal information structure remains a finite monotone partition. Section 6 concludes and discusses the limits of our analysis.

2 Model

2.1 Utility

A seller has a single object for sale. There are I potential bidders for the auction, indexed by $i \in \{1, ..., I\}$. Each agent i has a compact set $V_i = [0, 1]$ of possible valuations for the object, where a generic element is denoted by $v_i \in V_i$, and

$$V = \bigtimes_{i=1}^{I} V_i = [0, 1]^I .$$

We occasionally adopt the notation $v = (v_i, v_{-i})$. The valuation v_i is independently distributed with prior distribution function $F_i(v_i)$. The prior distribution function $F_i(v_i)$ is common knowledge. The associated density function $f_i(v_i)$ is positive on V_i. The utility of the (winning) agent is quasilinear and given by

$$u_i(v_i, t_i) = v_i - t_i,$$

where t_i is a monetary transfer.

2.2 Information Structure

The signal space is denoted by $S_i \subseteq [0, 1]$. The space S_i can either be countable, finite or infinite, or uncountable. Let $(V_i \times S_i, \mathcal{B}(V_i \times S_i))$ be a measurable space, where $\mathcal{B}(V_i \times S_i)$ is the class of Borel sets of $V_i \times S_i$. An *information structure* for agent i is given by a pair $\mathcal{S}_i \triangleq \langle S_i, F_i(v_i, s_i)\rangle$, where S_i is the space of signal realizations and $F_i(v_i, s_i)$ is a joint probability distribution over the space of valuations V_i and the space of signals S_i.[2] We refer to this class of information structures as (Borel) measurable information structures.

[2] By assumption, the signal of agent i is independent of agent j's valuation, for $j \neq i$. If agent i's signal were to depend on agent j's valuation, then full rent extraction is possible, see Cremer & McLean (1988).

The distribution and the information structure for all agents are denoted by omitting the subscript i, or $F(v, s)$ and $\mathcal{S}$, respectively. The joint probability distribution is defined in the usual way by

$$F_i(v_i, s_i) \triangleq \Pr(\widetilde{v}_i \leq v_i, \widetilde{s}_i \leq s_i).$$

The marginal distributions of $F_i(v_i, s_i)$ are denoted with minor abuse of notation by $F_i(v_i)$ and $F_i(s_i)$ respectively. For $F_i(v_i, s_i)$ to be part of an information structure requires the marginal distribution with respect to v_i to be equal to the prior distribution over v_i. The conditional distribution functions derived from the joint distribution function are defined in the usual way:

$$F_i(v_i |s_i) \triangleq \frac{\int_0^{v_i} dF_i(\cdot, s_i)}{\int_0^1 dF_i(\cdot, s_i)},$$

and similarly,

$$F_i(s_i |v_i) \triangleq \frac{\int_0^{s_i} dF_i(v_i, \cdot)}{\int_0^1 dF_i(v_i, \cdot)}.$$

The auctioneer can choose an arbitrary information structure $\mathcal{S}_i$ for every bidder i subject only to the restriction that the marginal distribution equals the prior distribution of v_i. The cost of every information structure is identical and set equal to zero. The choice of $\mathcal{S}_i$ is common knowledge among the bidders. At the interim stage every agent observes privately a signal s_i rather than her true valuation v_i of the object. Given the signal s_i and the information structure $\mathcal{S}_i$ each bidder forms an estimate about her true valuation of the object. The expected value of v_i conditional on observing s_i is defined and given by

$$w_i(s_i) \triangleq \mathbb{E}[v_i |s_i] = \int_0^1 v_i dF_i(v_i |s_i).$$

Every information structure $\mathcal{S}_i$ generates a distribution function $G_i(w_i)$ over posterior expectations given by

$$G_i(w_i) = \int_{\{s_i : w_i(s_i) \leq w_i\}} dF_i(s_i).$$

We denote by W_i the support of the distribution function $G_i(\cdot)$. Observe that the prior distribution $F_i(\cdot)$ and the posterior distribution over expected values $G_i(\cdot)$ need not coincide. For future discussions it is helpful to illustrate some specific information structures. The information structure $\mathcal{S}_i$ yields perfect information if $F_i(v_i) = G_i(v_i)$ for all $v_i \in V_i$.

In this case, the conditional distribution $F(s_i|v_i)$ has to satisfy

$$F_i(s_i|v_i) = \begin{cases} 0 & \text{if} \quad s_i < s(v_i), \\ 1 & \text{if} \quad s_i \geq s(v_i), \end{cases} \tag{1}$$

where $s(v_i)$ is an invertible function. An information structure which satisfies (1) without necessarily satisfying the invertibility condition is called *partitional.* An information structure is called *discrete* if S_i is countable and *finite* if S_i is finite.

After the choice of the information structures $\mathcal{S}_i$ by the auctioneer, the induced distribution of the agent's (expected) valuations is given by $G_i(w_i)$ rather than $F_i(v_i)$. The signal s_i and the corresponding expected valuation $w_i(s_i)$ remain private signals for every agent i and the auctioneer still has to elicit information by respecting the truthtelling conditions.

2.3 Mechanism

The seller selects the information structures of the bidders and a revelation mechanism. The objective of the seller is to maximize his expected revenue subject to the interim participation and interim incentive constraints of the agents. By the revelation principle we may restrict attention to the direct revelation mechanism. *The direct revelation mechanism* consists of a tuple $(W_i, t_i, q_i)_{i=1}^I$ with transfer payment of bidder i:

$$t_i : \mathop{\times}_{i=1}^{I} W_i \to \mathbb{R},$$

and the probability of winning the object for bidder i:

$$q_i : \mathop{\times}_{i=1}^{I} W_i \to [0, 1].$$

We sometimes write $T_i(w_i)$ for the expected transfer payment,

$$T_i(w_i) \triangleq \mathbb{E}_{w_{-i}} t_i(w_i, \cdot),$$

where the expectation is taken over $w_{-i} = (w_1, ..., w_{i-1}, w_{i+1}, ..., w_I)$. Similarly, $Q_i(w_i)$ denotes the expected probability of winning,

$$Q_i(w_i) \triangleq \mathbb{E}_{w_{-i}} q_i(w_i, \cdot).$$

The interim utility of bidder i with an expected valuation w_i and announced valuation $\widehat{w}_i$ is:

$$U_i\left(w_i, \widehat{w}_i\right) = w_i Q_i(\widehat{w}_i) - T_i(\widehat{w}_i).$$

The mechanism has to satisfy the *interim* participation constraints:

$$U_i\left(w_i\right) \triangleq U_i\left(w_i, w_i\right) \geq 0, \text{ for all } w_i \in W_i,$$

and the *interim* incentive constraints:

$$U_i\left(w_i\right) \geq U_i\left(w_i, \widehat{w}_i\right), \text{ for all } w_i, \widehat{w}_i \in W_i.$$

A mechanism that satisfies both, the interim participation constraints and the interim incentive constraints, is called incentive compatible. The timing of the events is graphically summarized below:

information structure $\mathcal{S}$, mechanism $\left(W_i, q_i, t_i\right)_{i=1}^I$, determined	$\left(s_i\right)_{i=1}^I$ realized $\left(w_i\right)_{i=1}^I$ observed	$\left(\widehat{w}_i\right)_{i=1}^I$ reported $\left(q_i\left(\cdot\right), t_i\left(\cdot\right)\right)_{i=1}^I$ assigned

Figure 0: Time line of events

We note that the transfers and the information structures are determined simultaneously for all bidders. In particular, we do not consider sequential mechanisms in which the information structure for some agents may be determined after some information has already been revealed about a certain subset of bidders.

3 Examples

This section illustrates properties of optimal information structures for some special cases. First, we look at single and two-bidder auctions. We illustrate the unconstrained optimal information structure. Then, we illustrate the constrained optimal information structure when the seller's choice is restricted to (i) identical information structures across bidders and (ii) identical partitions consisting of equally sized intervals. The examples illustrate that the seller prefers sparse information and treat bidders asymmetrically. Second, we depart from the two-bidder model and depict properties of the numerical solution to an auction with many bidders when the valuations are drawn from the uniform distribution. Again, we depict the unconstrained and constrained symmetric solution.

Single Bidder Auction: Consider first the case with a single bidder. The information structure in which the seller assigns a perfectly informative information structure to the bidder is analyzed in Myerson (1981). Myerson establishes that the seller can extract at most the virtual valuations in any incentive compatible selling mechanism. The virtual valuation of a bidder of type with valuation v equals the valuation of the bidder minus the incentive cost,

$$v - \frac{1 - F(v)}{f(v)}$$

Notice, that the incentive cost is positive and remains positive even if the seller assigns an information structure which informs partially only.

In contrast consider the situation in which the seller chooses to assign an uninformative information structure to the buyer. Without any information, a bidder is willing to pay up to the ex ante expected valuation of the bidder to receive the object. In this case, the seller can extract all the expected surplus. It is therefore immediate that assigning an uninformative information structure is optimal in a single bidder auction. The seller can post a price equal to the ex ante expected valuation. This posted price scheme extracts the total surplus *and* is efficient. Moreover, if the seller were to assign an information structure that informs the bidder, the seller would be worse off because he incurs an incentive cost expressed by the virtual utility.

Two-bidder Auction: Suppose now we were to add a second bidder to the auction with an identical prior distribution. The policy to disclose no information does not remain optimal with two bidders. To see this, notice that assigning an uninformative information structure extracts at most the ex ante expected valuation of the winning bidder. But with symmetric bidders, the revenue for the auctioneer would then be the same as in the case of a single bidder. In a two-bidder auction there is a simple scheme that achieves more rent by exploiting the increase in the number of bidders. The scheme has the following feature: The seller assigns an uninformative information structure to the first bidder as in the case of a single bidder auction, but assigns a binary information structure to the second bidder. A binary information structure permits the bidder to determine whether the valuation is above or below a certain threshold. The optimal threshold is exactly equal the ex-ante expected value of the object. The scheme then works as follows: Initially, the seller offers the object to the second bidder at a price equal to the conditional expected valuation in the event that the valuation is above the threshold. If the second bidder rejects the offer, then the seller offers the object to the first bidder at a price equal to the ex ante expected valuation. The total revenues to the seller under this scheme exceed the ex ante expected valuation of a bidder. Thus, the revenues under this scheme are higher than under a scheme in which the seller assigns an uninformative information structure. We observe that as before, the seller leaves no informational rent to the bidders. However, the allocation is not necessarily efficient anymore, as it could be that the first bidder has in fact a higher valuation for the object than the second bidder with the binary partition. However, the coarse information structure doesn't allow the seller to make this contingent decision.

In fact, it can be shown that the described information structure maximizes the revenues to the seller with two bidders and uniformly distributed valuations. Ignoring elements in the information structure which are associated with zero winning probability events, as we do throughout this paper, ensures that the described information structure is the unique solution. If attention is restricted to the class of information structures with finite partitions, then this result follows immediately from the first and second order conditions for optimally chosen partitions. Our results in the subsequent sections establish that the described scheme with two bidders is indeed optimal for the uniform distribution under general information structures even permitting non-partitional and non-finite information structures. For non-uniform prior distributions the optimal information structure may change as both, the location of the boundary points in the partition and the number of elements in the partition, depend on the distributional assumption.

The scheme with two bidders has a number of features that are worth emphasizing. First, even if bidders have initially symmetric prior distributions of valuations, they are optimally assigned asymmetric information structures. The first bidder receives no information, while the second bidder learns wether the valuation is above or below the ex-ante mean. Second, the seller does not give an informational rent to buyers. Both bidders are offered the object at a fixed price that they can accept or reject.

Symmetric Information Structures: Suppose the auctioneer were constrained to offer identical information structures to bidders. With two bidders and uniformly distributed valuations the binary nature of the informational structure remains optimal, but the location of the boundary point in the partition is altered by the symmetry restriction. It is now optimal to set the boundary point in the partition at one third and to offer the bidders the object at a fixed price of two third. If bidders valuations do not exceed one third the seller retains the object. The event of no award can occur because the cost of information revelation is high and offsets the gains from a sale when valuations are low.

Multiple Bidders: A natural question is whether the features of the optimal information structure for two bidders with uniformly distributed valuations extend to more general settings. We address this question in the subsequent sections. Before we start our formal analysis we illustrate graphically optimal information structures with many bidders.[3] The following figure depicts properties of optimal information structures with uniform distributed valuations as we vary the number of bidders. The dotted line illustrates the boundary points for constrained symmetric partitions. The solid line illustrates the boundary points for unconstrained (asymmetric) partition for the bidder with the largest interior boundary point.

INSERT FIGURE 1 HERE

As can be seen in the figure the number of boundary points increases monotone with the number of bidders participating in the auction. However, the increase is only very gradual. For the optimal (asymmetric) information structure, we count three elements in the partition with three to six bidders, four elements with seven to fifteen bidders and five elements with sixteen or more bidders. The boundary points of the partitions for the constrained optimal (symmetric) information structure look very similar to the unconstrained solution. We count three elements in the partition with four to eight bidders, four elements with nine to

[3] The numerical calculations were implemented using the software package GAUSS.

thirteen bidders and five elements with fourteen or more bidders.

Figure 1 illustrates further that in general it is not the case that the seller leaves no informational rent to the bidder. With three or more bidders, binary partitions are no longer optimal and as the auctioneer has to reward agents to report truthfully, he will have to incur incentive costs. As the number of bidders increases, the information structure becomes finer. The intuition is that with more competition the incentive costs due to the informational rents are lower and the revenue gains from improving allocative efficiency due to more information become more important, as the number of bidders increases.

4 Optimal Information Structure With Finite Signals

Motivated by the examples, this section describes the optimal auction *and* optimal information structure given a finite number of signals. In the next section, we then show that the characterization and optimality of finite information structure persists with an arbitrary number of signals, finite or infinite. Subsection 4.1 characterizes the seller's expected revenues and optimal mechanism for a *given* finite information structure and hence finite types. In Subsection 4.2 we start the analysis of the optimal information structure by deriving several features of the virtual utilities of the bidders. Subsection 4.3 uses the revenue structure of the optimal auction to show that the information structure has to be a partition and that an optimal information structure for a given finite number of signals exists.

4.1 Optimal Auction Design With Finite Types

Motivated by the examples, this section characterizes the seller's expected revenues and optimal mechanism for a given finite information structure and hence finite types. At this stage we are merely interested in characterizing the expected revenues of the auctioneer from bidder i. For a given distribution $G_i(w_i)$, we denote the finite set of mass points by $\left\{w_i^1, ..., w_i^K\right\}$, and for every w_i^k,

$$g_i^k \triangleq G_i\left(w_i^k\right) - G_i\left(w_i^{k-1}\right) > 0,$$

with g_i^k being the positive probability of mass point w_i^k. For notational ease, we shall denote the value of the distribution function $G_i(\cdot)$ at w_i^k simply as G_i^k, and likewise refer to the interim probability of winning at w_i^k as Q_i^k and the interim transfer at w_i^k as T_i^k. Lemma 1 describes the revenues the auctioneer receives from bidder i with a given prior distribution $G_i(\cdot)$ and a given expected probability of winning $Q_i(\cdot)$.

Lemma 1 (Revenues)

The expected revenues from bidder i in an incentive compatible mechanism are:

$$R_i(G_i, Q_i) \triangleq \sum_{k=1}^{K} \left[w_i^k - \left(w_i^{k+1} - w_i^k \right) \frac{1 - G_i^k}{g_i^k} \frac{\widetilde{Q}_i^k}{Q_i^k} \right] Q_i^k g_i^k - U_i\left(w_i^1\right), \tag{2}$$

subject to $Q_i(\cdot)$ *being non-decreasing,* $Q_i^k \leq \widetilde{Q}_i^k \leq Q_i^{k+1}$ *and* $U_i\left(w_i^1\right) \geq 0$.

Proof. The proofs for all results are provided in the appendix. ■

The similarity with the case of positive density analyzed in Myerson (1981) is immediate. The modification due to the discreteness appears in the obvious places. The density $g_i\left(w_i^k\right)$ is now replaced with the positive probability g_i^k. The local change $dw_i = 1$ is being replaced by the discrete change between w_i^k and w_i^{k+1}, or $w_i^{k+1} - w_i^k$.

There are two indeterminacies in the expression of revenues (2). First, as in the continuous analogue, the utility for the lowest type, $U_i\left(w_i^1\right)$, is an arbitrary non-negative number. Second, the probability $\widetilde{Q}_i^k$ is an arbitrary number in $\left[Q_i^k, Q_i^{k+1}\right]$. The second indeterminacy arises due to the discreteness of types and is absent in the continuous analogue. With discrete types, the utility increment for a bidder of type w_i^k attributable to the (hypothetical) gain of mimicking the adjacent lower type can be weighed with probability Q_i^k or Q_i^{k+1}. In fact, any probability $\widetilde{Q}_i^k$ contained in $\left[Q_i^k, Q_i^{k+1}\right]$ yields incentive compatible revenues.

Henceforth, we select $\widetilde{Q}_i^k = Q_i^k$ and $U_i\left(w_i^1\right) = 0$. This choice maximizes the seller's revenue for given (G_i, Q_i). Since we seek the information structure and mechanism that maximizes seller's revenues, we can make this selection without loss of generality. This leads us to the following expression for seller's revenues:

$$R_i(G_i, Q_i) = \sum_{k=1}^{K} \left[w_i^k - \left(w_i^{k+1} - w_i^k \right) \frac{1 - G_i^k}{g_i^k} \right] Q_i^k g_i^k.$$

The associated interim transfers of agent i satisfy the incremental relationship:

$$T_i^{k+1} = T_i^k + \left(Q_i^{k+1} - Q_i^k \right) w_i^{k+1}, \tag{3}$$

and the expected revenues from agent i can alternatively be represented as

$$R_i\left(G_i, Q_i\right) = \sum_{k=1}^{K} g_i^k \left[\sum_{l=1}^{k} \left(Q_i^l - Q_i^{l-1}\right) w_i^l\right], \tag{4}$$

with the convention that $Q_i^0 = 0$.

The revenues of the auctioneer from bidder i are characterized as a function of the expected probability of winning $Q_i\left(w_i\right)$ with a value w_i. The interaction with the valuation of the other bidders is represented by expectations over the valuations w_{-i}. Now, we disaggregate the expression and consider the dependence on the realizations of all valuations explicitly. The revenue of the auctioneer from all bidders is given by:

$$R\left(G, q\right) \triangleq \sum_{k_1=1}^{K_1} \cdots \sum_{k_I=1}^{K_I} \left[\sum_{i=1}^{I} q_i\left(w_1^{k_1}, ..., w_I^{k_I}\right) \left[w_i^{k_i} - \left(w_i^{k_i+1} - w_i^{k_i}\right) \frac{1 - G_i^{k_i}}{g_i^{k_i}}\right] \prod_{i=1}^{I} g_i^{k_i}\right], \tag{5}$$

where $q_i\left(w\right) \geq 0$ and $\sum_{i=1}^{I} q_i\left(w\right) \leq 1$. The optimal auction is then given by the probability vector $q\left(w\right) = \left(q_1\left(w\right), ..., q_I\left(w\right)\right)$ which maximizes the expected revenue (5). Define the virtual utility with discrete types by:

$$\gamma_i^k \triangleq w_i^k - \left(w_i^{k+1} - w_i^k\right) \frac{1 - G_i^k}{g_i^k}.$$

If the virtual utilities are monotone, then the optimization problem can be solved pointwise, i.e. for any type realization $w = \left(w_1^{k_1}, w_2^{k_2}, ..., w_I^{k_I}\right)$ by solving

$$\max_{\{q_i(w)\}_{i=1}^{I}} \sum_{i=1}^{I} q_i\left(w_1^{k_1}, ..., w_I^{k_I}\right) \left[w_i^{k_i} - \left(w_i^{k_i+1} - w_i^{k_i}\right) \frac{1 - G_i^{k_i}}{g_i^{k_i}}\right]$$

subject only to the familiar restriction that $q_i\left(w\right) \geq 0$ and $\sum_{i=1}^{I} q_i\left(w\right) \leq 1$. This pointwise optimization becomes possible as the monotonicity of virtual utilities guarantees the monotonicity of the interim winning probabilities $Q_i\left(w_i^k\right)$ as a function of w_i^k. We can now readily describe some properties of the optimal auction.

Corollary 1

Suppose the virtual utilities are increasing for every agent. The optimal auction is described by:

1. $\max\left\{\gamma_1^{k_1}, ..., \gamma_I^{k_I}\right\} > 0 \Rightarrow \sum_{i=1}^{I} q_i\left(w_1^{k_1}, ..., w_I^{k_I}\right) = 1$;

2. $q_i\left(w_1^{k_1},...,w_I^{k_I}\right) > 0 \Rightarrow \gamma_i^{k_i} \geq 0 \wedge \gamma_i^{k_i} \geq \gamma_j^{k_j}$, $\forall j$;

3. $q_i\left(w_1^{k_1},...,w_i^{k_i},...,w_I^{k_I}\right) > 0 \Rightarrow \forall w_i^{k_i'} > w_i^{k_i},\ q_i\left(w_1^{k_1},...,w_i^{k_i'},...,w_I^{k_I}\right) = 1.$

The characterization is the exact discrete type analog to the celebrated optimal auction result for 'regular environments' by Myerson (1981) with a continuum of types. If the virtual utilities γ_i^k for a given distribution function G_i^k were not monotone, then the optimal auction would be subject to a similar "ironing out" procedure as necessary in an optimal auction with a continuum of types. We conclude the section with a partial characterization of the "ironing out" procedure for future reference.

Corollary 2
The optimal mechanism satisfies for all $\gamma_i^k, \gamma_i^{k+1}$ *with* $\gamma_i^k > \gamma_i^{k+1}$*:* $Q_i^k = Q_i^{k+1}$.

4.2 Virtual Utilities

We first argue that the optimal information structure will always generate virtual utilities which are strictly increasing. Recall the basic incentive compatibility condition for any Bayesian implementable auction is that the winning probability Q_i^k is increasing in the valuation w_i^k. The revenue formula (5) on the other hand implies that the winning probability Q_i^k is increasing in the virtual utility γ_i^k of the agent i. If the virtual utilities γ_i^k generated by a given distribution G_i^k were not monotone, then the optimal auction would be subject to an "ironing out". The basic element in the former procedure is to maintain the expected probability Q_i^k constant over a set of types which covers the non-monotonicity in the virtual utilities. As the constant probability essentially implies that the incentives and revenues are also constant on the set, the question arises as to whether the auctioneer has any interest in distinguishing between different types in this set. In fact, as the information structure is chosen by the auctioneer, he may wish to bundle types to which identical allocations have to be offered in any case. In other words, when the auctioneer can choose the information structure for the bidders, the "ironing out" of non-monotonicities in the virtual utility may be achieved by a sufficient coarsening of the information structure rather than through constant winning probabilities of the form: $Q_i^k = Q_i^{k+1}$. The consequence of this argument leads to the next result.

Lemma 2 (Monotone Virtual Utilities)

The optimal virtual utilities are strictly increasing.

By Lemma 2, we can describe the set of optimal virtual utilities for bidder i by an ordered set $\Gamma_i = \{\gamma_i^1, ..., \gamma_i^k, ..., \gamma_i^K\}$, with $\gamma_i^1 < \gamma_i^2 < ... < \gamma_i^K$. The local argument regarding the benefits of a coarser information structure has some additional implications for the structure of the set of virtual utilities Γ_i. Consider two adjacent and positive virtual utilities by agent i, say γ_i^k and γ_i^{k+1}. Suppose now that these two virtual utilities do not bracket any virtual utility by a competitor, or more precisely that

$$\left\{ \gamma_j \,\middle|\, \gamma_i^k < \gamma_j < \gamma_i^{k+1}; \gamma_j \in \Gamma_j, j \neq i \right\} = \emptyset. \tag{6}$$

By Corollary 1, the virtual valuations γ_i^k and γ_i^{k+1} would then win against the same type realizations of the competitors and in turn they would receive the object with the same probability: $Q_i^k = Q_i^{k+1}$. But then we can use precisely the argument of Lemma 2 to conclude that a coarser information structure would increase the revenues of the auctioneer.

Lemma 3 (Adjacent and Asymmetric Virtual Utilities)

1. *For* $\forall i, \forall k < K : \left\{ \gamma_j \,\middle|\, \gamma_i^k < \gamma_j < \gamma_i^{k+1}; \gamma_j \in \Gamma_j, j \neq i \right\} \neq \emptyset.$
2. $\exists i, j$ *such that* $\Gamma_i \neq \Gamma_j$.

A direct consequence of the alternating structure of the virtual utilities is the asymmetry of the virtual utilities indicated by the second part of Lemma 3. With two bidders, the same argument leads immediately to a stronger result, namely that $\Gamma_i \cap \Gamma_j = \emptyset$. With more than two bidders, our argument does not preclude the possibility that some bidders may have virtual utilities in common.

The asymmetry of the virtual utilities implies asymmetry of the information structure even if the underlying distributions over valuations are symmetric. For legal or fairness reasons, symmetric treatment of bidders may be a requirement in the auction. It is worth emphasizing that if we impose a symmetry requirement on the information structure, then the basic properties of the optimal symmetric information structure will qualitatively remain identical to the ones without the symmetry requirement.[4]

4.3 Monotone Partitions

A partitional information structure can be represented without recourse to a joint distribution over the space of valuations and signals by a partition of the original space V_i. A partition is a collection of subsets, with slight abuse of notation, denoted by $\mathcal{S}_i = \left\{S_i^k\right\}$ such that for all k, k' we have $S_i^{k'} \cap S_i^k = \emptyset$ and

$$\bigcup_{k=1}^{K} S_i^k = V_i.$$

The partition is *monotone* if for any $v_i, v_i' \in S_i^k$, $\lambda v_i + (1-\lambda) v_i' \in S_i^k$ for all $\lambda \in [0,1]$.

Theorem 1 (Monotone Partition)

1. *For every fixed $K < \infty$, an optimal information structure exists.*

2. *The optimal information structure is a monotone partition.*

The result that the optimal information structure is a partition as well as the monotonicity of the partition itself stem from the same elementary argument based on a necessary condition of optimality. The argument is local in the sense that we hold the information structures and conditional winning probabilities of other bidders constant and look only at the revenues to the auctioneer from bidder i. The focus on the single agent i allows us to illustrate the result with a simple diagram, which represents the incentive compatible revenues from bidder i. The diagram depicts the valuations w_i of agent i on the x-axis and the interim probabilities Q_i on the y-axis. In the diagram every rectangle of surface $w_i^k Q_i^k$ represents the gross social surplus generated by type w_i^k with the winning probability Q_i^k determined by the optimal auction. We showed in Section 4.1 that the interim incentive compatible transfers satisfy the relationship:

$$T_i^{k+1} = T_i^k + \left(Q_i^{k+1} - Q_i^k\right) w_i^{k+1}. \tag{7}$$

[4] The earlier example illustrates the similarity of the optimal information structure with and without the symmetry requirement.

The horizontal rectangles in Figure 2a represent the share of marginal surplus from the next higher type which goes to the auctioneer and the vertical rectangle represents the share which goes to agent i. Notably absent from the diagram are the probabilities g_i of agent i and indeed the interim transfer payments T_i are independent of g_i. >From the diagram, we can infer several general properties of the optimal auction. First, the social surplus is increasing in w_i^k and this property is shared by the indirect utility of the auctioneer and the agent. Second, while there is genuine sharing of the surplus, the sharing rule is not linear and depends on the winning probabilities determined by the optimal auction.

Insert Figure 2 Here

The optimality of a given information structure requires that the auctioneer does not wish to introduce further randomization into the information structure. A specific and local version of such a randomization can be represented as a mass preserving mixture between two adjacent expected valuations, w_i^k and w_i^{k+1}, which is given by the following modification:

$$w_i^k(\varepsilon) = \frac{\left(g_i^k - \varepsilon\right) w_i^k + \varepsilon w_i^{k+1}}{g_i^k}, \tag{8}$$

and

$$w_i^{k+1}(\varepsilon) = \frac{\left(g_i^{k+1} - \varepsilon\right) w_i^{k+1} + \varepsilon w_i^k}{g_i^{k+1}}, \tag{9}$$

for some ε satisfying, $0 < \varepsilon \leq g_i^k, g_i^{k+1}$. Clearly, we can find a signal structure and joint distribution to generate the expected valuations for every ε. The effect of a positive ε is depicted in Figure 2b. It increases w_i^k and the marginal revenue from type k, but decreases w_i^{k+1} and likewise the marginal revenue from type $k+1$. By mixing, we understand here that we associate (via the signals) low true valuations with high expected valuations, and conversely high true valuations with low expected valuations.

Suppose now that the optimal information structure (and auction) requires $\varepsilon = 0$. In consequence an increase in ε would decrease the revenues. With the local changes as suggested by (8) and (9) the marginal revenue as a function of ε is linear as can be immediately inferred from the incentive compatible revenue representation:

$$\sum_{k=1}^{K} g_i^k \left[\sum_{l=1}^{k} \left(Q_i^l - Q_i^{l-1} \right) w_i^l \right],$$

as we keep the conditional probabilities Q_i^k and type probabilities g_i^k constant. The argument for a monotone partition is now based on the following idea. Suppose an optimal information structure is not a monotone partition. Then by the first-order conditions further mixing would decrease the revenues. But the same conditions also allow us to infer the converse. Further de-mixing would increase the revenues. As every information structure which is not a monotone partition presents the possibility of some de-mixing between at least two adjacent types, this demonstrates the optimality of a monotone partition.

5 Optimal Information Structure without Finite Signals

So far we have obtained a number of qualitative results for optimal information structures when the signal space of each agent contained at most K elements. In this section we establish that the optimal information structure is indeed finite and monotone partition in the class of all measurable information structures as defined in Section 2.

Consider any incentive compatible mechanism (q, t) and the distribution G over expected valuations induced by any arbitrary information structure. Types in the distribution G can have zero density, positive density, or positive probability.

Proposition 1 (Approximation)
Let G be a distribution generated by an arbitrary information structure $\mathcal{S}$ and let (q, t) be an incentive compatible mechanism. For any $\varepsilon > 0$ there exists a distribution function $\widehat{G}$ generated by a finite information structure $\widehat{\mathcal{S}}$ and an incentive compatible mechanism $\left\{\widehat{q}, \widehat{t}\right\}$ such that

$$R\left(\widehat{G}, \widehat{q}\right) \geq R\left(G, q\right) - \varepsilon$$

Proposition 1 establishes that the set of revenues generated by any incentive compatible mechanism with a finite information structure is dense in the set of revenues generated by incentive compatible mechanism with an arbitrary information structure. Hence the incentive compatible revenues generated by an arbitrary information structure can be approximated arbitrarily well by a finite information structure.[5]

[5] We would like to thank an anonymous referee for suggesting this continuity result.

A similar local variation allows us to establish an important property of the conditional winning probabilities Q_i. This property will play a central role in the argument to demonstrate that a finite information structure is optimal.

Lemma 4 (Increasing Differences)
The conditional winning probabilities Q_i^k satisfy strictly increasing differences.

Lemma 4 establishes that the conditional winning probabilities of any bidder have the property of strictly increasing differences, or that

$$Q_i^{k+1} - Q_i^k > Q_i^k - Q_i^{k-1}.$$

Henceforth we shall refer to this property for simplicity as the convexity of the conditional winning probabilities even though they are defined over a finite set of indices.[6] The proof of the above Lemma relies again on a local argument as we examine the revenue from bidder i only.

Theorem 2 (Existence)
An optimal information structure in the class of all Borel measurable information structures exists and it is a finite monotone partition.

The proof of Theorem 2 proceeds in three steps: (1) an optimal information structure and associated revenues exist in the class of finite information structures; (2) the revenues from the optimal finite mechanism are maximal in the class of all discrete (possibly non-finite) information structures; and (3) the finite information structure revenues are also maximal in the class of all measurable information structures. Theorem 2 builds immediately on our earlier results: By Proposition 1, we can restrict attention to sequences of mechanisms with finite information structures instead of arbitrary information structures. By Theorem 1, for every finite K a solution exists. If we consider any sequence, then by

[6] We chose to first establish properties of the virtual utilities in Lemma 2 and 3, and then use these properties to derive the partition property of the information structure and increasing differences of the winning probabilities. Alternatively, we could start by establishing the increasing difference property and then proceed to virtual utilities and the partition property.

Lemma 4 any element of the sequence must have convex conditional winning probabilities for every bidder. It follows that the limiting values have to be convex as well. Consider now the limit of the conditional winning probability of the type of agent i with the lowest strictly positive virtual utility for every finite K. If the limit $\overline{Q}_i^1$ is positive, then by Lemma 4 we can conclude that at most $1/\overline{Q}_i^1$ signals can have a positive probability. In particular, the convexity of the conditional winning probability allows us to assert that the conditional winning probabilities of agent i have to satisfy for all i and k

$$\overline{Q}_i^k - \overline{Q}_i^{k-1} \geq \overline{Q}_i^1,$$

and thus the optimal information structure has to be finite. If the limit $\overline{Q}_i^1$ is zero, then the argument is a little more subtle. Essentially, we use the fact of $\overline{Q}_i^1 = 0$ to show that there exists at least one agent j whose lowest type has strictly positive probability $\overline{g}_j^1$ and strictly positive probability of winning $\overline{Q}_j^1$. This in turn allows us to show that at most a finite number of types of agent i can win and win in particular against $\overline{w}_j^1$. We are thus lead to conclude that the optimal information structure exists, is finite, and by Theorem 1 it has to be a monotone partition.

The joint optimality of discrete information structures and convex winning probabilities is now illustrated using the following informal reasoning, based on well-known results for continuous rather than discrete types. Consider again the interim problem with a single bidder i. The social surplus from type v_i is given by $v_i Q_i(v_i)$. We know that the marginal indirect utility of type v_i in an incentive compatible mechanism is $Q_i(v_i)$. The residual marginal gains $v_i Q_i'(v_i)$ consequently belong to the auctioneer. It further follows that the indirect utility of the agent is convex as $Q_i(v_i)$ is increasing and that the social surplus as well as the auctioneer's surplus is convex if $Q_i(v_i)$ is not too concave. The auctioneer receives from agent i the expected revenue given by:

$$\int_0^1 \left[\int_0^{v_i} r_i Q_i'(r_i)\, dr_i \right] dF_i(v_i). \tag{10}$$

The single bidder scenario suppresses the decision as to how large $Q_i(v_i)$ should be. This will be naturally determined by the opportunity cost stemming from allocating the object to the competing bidders. We now pursue the following thought experiment. Suppose the auctioneer had decided to give a small interval of types, say $[\underline{v}_i, \overline{v}_i]$ a fixed aggregate probability, say $\widehat{Q}$, with

$$\int_{\underline{v}_i}^{\overline{v}_i} Q_i(v_i)\, dF_i(v_i) = \widehat{Q}, \tag{11}$$

and all he had to decide is how to allocate this total probability inside the interval. If further he were only concerned with maximizing the surplus that he can extract from all higher types, then he should pursue the following objective function:

$$\max_{Q_i(v_i)} \int_{\underline{v}_i}^{\overline{v}_i} v_i Q_i'(v_i)\, dF_i(v_i),$$

subject to the constraint (11), to maximize the integral of marginal gains. As the marginal increment in $Q_i(v_i)$ is weighed by v_i, it is then easily seen that the auctioneer would indeed like to choose a very convex function for the winning probabilities as the marginal increases $Q_i'(v_i)$ would then receive the largest possible weight. However this exclusive concern with the marginal revenue is mitigated in the 'standard' optimal auction by the concern for the inframarginal revenue as represented by the complete revenue function (10). However, by controlling the information structure we can bundle types together to (locally) eliminate the inframarginal concern and pursue only the maximization of the marginal revenues.

6 Discussion

This paper reconsidered the design of the optimal auction by making the information structure an integral part of the design problem. Notable features of the optimal information structure were the partitional character, the finiteness of the partition and therefore of private types as well, and the asymmetry of the information structures. The analysis reveals an important trade-off between the minimization of information rent and the maximization of allocational efficiency. The optimal information structure balances these two conflicting objectives.

We would like to emphasize that the current results may not inform us directly about auction design in practice.[7] While we expect the trade-off between information rent and

[7] There are many auctions in which the precision of the information available to the buyers is at least partially controlled by the seller. In US offshore *wildcat* oil tract auctions, the bidding firms are permitted to

allocational efficiency to remain important, the current analysis makes a number of assumptions which would have to be weakened to provide a better fit with empirical observations. We briefly discuss the restrictions imposed by the three key assumptions of the model: (i) the seller has complete control over the precision of each bidder's signal, (ii) each bidder is initially uninformed and (iii) the seller cannot price the information directly (through ex ante payments).

In the model, the seller is free to choose from the set of all information structures, and in particular, the seller can choose to leave the bidder uninformed about her true valuation. However, in practice the seller may be severely constrained in his choice of the information structure. For example, the information structure might be restricted to a noisy sampling process as in the offshore oil tract auctions, where the choice of informativeness is determined by the number of samples. In addition, each bidder may have some private information and thus leaving the bidder uninformed may not be a feasible information structure.

In our analysis, the seller offers allocations and prices only after each bidder has received her private signal. In particular, the seller cannot price the information structure directly. This assumption might be justified in light of the observable lack of direct pricing of information in auctions, as in the auctions mentioned in the above footnote. Yet, from a theoretical point of view there might be a tension between the ability to control the information structure and an inability to price the information structure. In fact, Eso & Szentes (2007) and Gershkov (2002) consider a similar setting to the one presented here, but allow the seller to price the information. Gershkov (2002) shows that the optimal solution then consists of participation fees equal to the expected bidders' rent followed by a

gather information about the lease value and their drilling costs prior to the sale using seismic information, but no on-site drilling is allowed. In contrast, in US offshore *drainage* oil leases, some bidders are intentionally given access to superior information by allowing them prior drilling in the area, see Porter (1995). Similarly, Genesove (1993) reports that in wholesale used car auctions, different auctioneers adopt strikingly different rules as to how potential bidders may inspect a used car before they place a bid on it. Auctions in which the seller intentionally limits the amount of information are sometimes referred to as "blind auctions" and documented examples are the licensing procedure for motion pictures, see Kenney & Klein (1983) and Blumenthal (1988), and the competition of brokers for the trade of a large portfolio on behalf of an institutional asset manager, see Kavajecz & Keim (2005) and Foucault & Lovo (2003).

standard Vickrey auction. In Eso & Szentes (2007) each bidder receives an initial private signal and possibly a second signal that can be released by the seller later on. They show that the seller can extract the rent associated with the signal released by the seller, but cannot extract the rent associated with the initial private signal. The empirical absence of a price for information in auctions suggests that additional factors might be at work. In a richer environment, the optimal information structure will then have to incorporate these factors. The basic trade-off analyzed here would then be augmented, but also rendered more complex by the nature of the constraints.

7 Appendix

The appendix contains the proofs to all lemmata, propositions and theorems in the text.

Proof of Lemma 1. The proof consists of two arguments: First, we establish a bound on the utility difference of two adjacent types, w_i^{k-1} and w_i^k as a function of the expected probability of winning Q_i^{k-1} and Q_i^k. Second, we use the bound repeatedly to obtain an expression for the expected transfer payment and thus revenue from bidder i. Along the way we shall show that the expected probability of winning $Q_i(\cdot)$ is non-decreasing.

Incentive compatibility requires that the allocation $\{Q_i(w_i), T_i(w_i)\}$ satisfies the interim incentive and participation constraints. The incentive constraint for a bidder w_i^k mimicking a bidder with expected valuation w_i^{k-1} yields:

$$U_i\left(w_i^k\right) = w_i^k Q_i^k - T_i^k \geq w_i^k Q_i^{k-1} - T_i^{k-1}. \tag{12}$$

Similarly, for bidder w_i^{k-1} who considers mimicking a bidder with expected valuation w_i^k yields:

$$U_i\left(w_i^{k-1}\right) = w_i^{k-1} Q_i^{k-1} - T_i^{k-1} \geq w_i^{k-1} Q_i^k - T_i^k. \tag{13}$$

Now, subtracting (13) from (12) yields the following set of inequalities:

$$\left(w_i^k - w_i^{k-1}\right) Q_i^k \geq U_i\left(w_i^k\right) - U_i\left(w_i^{k-1}\right) \geq \left(w_i^k - w_i^{k-1}\right) Q_i^{k-1}, \tag{14}$$

which gives bounds on the utility difference of two adjacent types, w_i^{k-1} and w_i^k, as a function of the expected probability of winning Q_i^{k-1} and Q_i^k. We observe that the outer inequality in (14) requires that:

$$\left(w_i^k - w_i^{k-1}\right)\left(Q_i^k - Q_i^{k-1}\right) \geq 0,$$

which implies that $Q_i(\cdot)$ is non-decreasing. Observe also that the interim participation constraint implies that $U_i\left(w_i^1\right) \geq 0$.

Next, we repeatedly apply the inequality in (14) to obtain an expression for the expected transfer payment and ultimately the revenue expression (2). An indeterminacy arises as the utility gain based on mimicking the adjacent lower type can be weighed with the left or the right-hand side probability, or any number in between. Accounting for this indeterminacy, the expression for the equilibrium utility equals:

$$U_i\left(w_i^k\right) = U_i\left(w_i^1\right) + \sum_{l=2}^{k}\left(w_i^l - w_i^{l-1}\right)\widetilde{Q}_i^k,$$

where $\widetilde{Q}_i^{l-1} \in \left[Q_i^{l-1}, Q_i^l\right]$ accounts for the indeterminacy. By definition, $T_i^k = w_i^k Q_i^k - U_i\left(w_i^k\right)$, and the expression for the expected transfer payment is given by:

$$T_i\left(w_i^k\right) = w_i^k Q_i^k - U_i\left(w_i^1\right) - \sum_{l=2}^{k}\left(w_i^l - w_i^{l-1}\right)\widetilde{Q}_i^{l-1},$$

which takes again the indeterminacy into account. The seller's revenues are obtained by the equivalent of integration by parts for the discrete probabilities. Doing so, leads to the formula:

$$R_i\left(G_i, Q_i\right) = \sum_{k=1}^{K}\left[w_i^k - \left(w_i^{k+1} - w_i^k\right)\frac{1 - G_i^k}{g_i^k}\frac{\widetilde{Q}_i^k}{Q_i^k}\right]Q_i^k g_i^k - U_i\left(w_i^1\right).\blacksquare$$

Proof of Corollary 1. The characterization follows immediately from pointwise optimization of the objective function (5) for any realization of values $w = \left(w_1^{k_1}, ..., w_I^{k_I}\right)$.■

Proof of Corollary 2. Suppose to the contrary (and by Lemma 1) that $Q_i^k < Q_i^{k+1}$. Then there must exist w_{-i} such that $q_i\left(w_i^k, w_{-i}\right) < q_i\left(w_i^{k+1}, w_{-i}\right)$. The incentive compatibility conditions of all agents except i, and in particular their conditional winning probabilities remain constant under $q_i(\cdot)$ and a modified probability assignment $\widehat{q}_i(\cdot)$ as long as

$$g_i^k q_i\left(w_i^k, w_{-i}\right) + g_i^{k+1} q_i\left(w_i^{k+1}, w_{-i}\right) = g_i^k \widehat{q}_i\left(w_i^k, w_{-i}\right) + g_i^{k+1}\widehat{q}_i\left(w_i^{k+1}, w_{-i}\right). \quad (15)$$

By the hypothesis of $\gamma_i^k > \gamma_i^{k+1}$, any $\hat{q}_i(\cdot)$ such that (15) is maintained and displays $q_i\left(w_i^k, w_{-i}\right) < \widehat{q}_i\left(w_i^k, w_{-i}\right)$ must strictly increase the revenues of the auctioneer, which delivers the contradiction.■

Proof of Lemma 2. Suppose to the contrary and hence that there exists γ_i^k and γ_i^{k+1} such that $\gamma_i^k \geq \gamma_i^{k+1}$. Suppose initially that indeed $\gamma_i^k > \gamma_i^{k+1}$. Then by Corollary 2, it follows that $Q_i^k = Q_i^{k+1}$. In contrast, consider the revenues from agent i if the original information structure were modified by sending a single signal $\widehat{s}_i$ whenever the original information structure emitted the signal s_i^k or s_i^{k+1}. The so modified information structure effectively joins the types w_i^k and w_i^{k+1} into a single type, denoted by $\widehat{w}_i$. The probability of the newly created type is given by

$$\widehat{g}_i = g_i^k + g_i^{k+1},$$

and its conditional expected value is:

$$\widehat{w}_i = \frac{w_i^k g_i^k + w_i^{k+1} g_i^{k+1}}{g_i^k + g_i^{k+1}}.$$

The difference in the revenue between the original and modified information structure is given, after some initial cancellations, by

$$\begin{aligned}
&R_i\left(\widehat{G}_i, Q_i\right) - R_i\left(G_i, Q_i\right) = g_i^k\left[\left(\widehat{Q}_i - Q_i^{k-1}\right)\widehat{w}_i - \left(Q_i^k - Q_i^{k-1}\right)w_i^k\right] \\
&+g_i^{k+1}\left[\left(\widehat{Q}_i - Q_i^{k-1}\right)\widehat{w}_i - \left(Q_i^k - Q_i^{k-1}\right)w_i^k - \left(Q_i^{k+1} - Q_i^k\right)w_i^{k+1}\right] \\
&+\sum_{l=k+2}^{K} g_i^l \left\{ \begin{array}{c} \left[\left(\widehat{Q}_i - Q_i^{k-1}\right)\widehat{w}_i + \left(Q_i^{k+2} - \widehat{Q}_i\right)w_i^{k+2}\right] \\ -\left[\left(Q_i^k - Q_i^{k-1}\right)w_i^k + \left(Q_i^{k+1} - Q_i^k\right)w_i^{k+1} + \left(Q_i^{k+2} - Q_i^{k+1}\right)w_i^{k+2}\right] \end{array} \right\}.
\end{aligned}$$

The combination of w_i^k and w_i^{k+1} affects only the revenue from all types starting at k. By construction, the conditional winning probability of the new type satisfies $\widehat{Q}_i = Q_i^k = Q_i^{k+1}$, and thus the difference simplifies to

$$\begin{aligned}
&R_i\left(\widehat{G}_i, Q_i\right) - R_i\left(G_i, Q_i\right) = g_i^k\left(Q_i^k - Q_i^{k-1}\right)\left(\widehat{w}_i - w_i^k\right) \\
&+g_i^{k+1}\left(Q_i^{k+1} - Q_i^{k-1}\right)\left(\widehat{w}_i - w_i^k\right) + \sum_{l=k+2}^{K} g_i^l\left\{\left(\widehat{Q}_i - Q_i^{k-1}\right)\left(\widehat{w}_i - w_i^k\right)\right\},
\end{aligned}$$

but by hypothesis, $w_i^{k+1} > w_i^k$, and hence $\widehat{w}_i - w_i^k > 0$, and thus each of the three terms are positive, yielding the desired result. Finally, in the case that $\gamma_i^k = \gamma_i^{k+1}$, there are several optimal solution for Q_i^k and Q_i^{k+1}, but since $Q_i^k = Q_i^{k+1}$ is always guaranteed to be one of them, the same argument goes through for the case of $\gamma_i^k = \gamma_i^{k+1}$.■

Proof of Lemma 3. (1.) Suppose to the contrary. Then there exist γ_i^k such that

$$\left\{\gamma_j \,\middle|\, \gamma_i^k < \gamma_j < \gamma_i^{k+1}, j \neq i\right\} = \emptyset.$$

Observe next that if two adjacent virtual utilities belong to bidder i then the probability of receiving the good has to be identical on both intervals, $Q_i^k = Q_i^{k+1}$ by Lemma 2 and Corollary 1. But by the same argument as Lemma 2, we may then join the mass points w_i^k and w_i^{k+1} and the expected revenues for the auctioneer will strictly increase, a contradiction.

(2.) Suppose to the contrary and thus $\Gamma_i = \Gamma_j$ for all i, j. Then there exists *an* optimal auction such that for some i and some k, $Q_i^k = Q_i^{k+1}$. We can now appeal to the same argument as in (1.) to conclude that the revenues of the auctioneers can be strictly increased by joining the mass points w_i^k and w_i^{k+1}, which destroys the symmetry in the virtual utilities.■

Proof of Theorem 1. We first establish that there is always a monotone partition which achieves strictly higher revenues than any other finite information structure. We then argue that a optimal monotone partition exists.

(2.) A necessary condition for an optimal information structure is that given the type probabilities g_i^k and the winning probabilities Q_i^k, the auctioneer does not wish to introduce further randomization into the information structure. A local version of such a randomization is a mass preserving mixture between w_i^k and w_i^{k+1}. If w_i^k and w_i^{k+1} are candidate types, then a local randomization between these two types is given by the following modification:

$$w_i^k(\varepsilon) \triangleq \frac{\left(g_i^k - \varepsilon\right) w_i^k + \varepsilon w_i^{k+1}}{g_i^k}, \tag{16}$$

and

$$w_i^{k+1}(\varepsilon) \triangleq \frac{\left(g_i^{k+1} - \varepsilon\right) w_i^{k+1} + \varepsilon w_i^k}{g_i^{k+1}}, \tag{17}$$

for some ε satisfying, $0 < \varepsilon \leq g_i^k, g_i^{k+1}$. We denote the revenue resulting from the modification as a function of ε by $R_i(\varepsilon | G_i, Q_i)$ for given G_i^k and Q_i^k. A necessary condition for the optimality of the information structure is

$$R_i'(0 | G_i, Q_i) \leq 0. \tag{18}$$

The function $R_i(\varepsilon | G_i, Q_i)$ is linear in ε and the derivative $R_i'(\varepsilon | G_i, Q_i)$ can be written as:

$$\begin{aligned} R_i'(\varepsilon | G_i, Q_i) &= \\ \left(w_i^{k+1} - w_i^k\right) \frac{\left(Q_i^k - Q_i^{k-1}\right) g_i^{k+1}\left(1 - G_i^{k-1}\right) - \left(Q_i^{k+1} - Q_i^k\right) g_i^k \left(1 - G_i^k\right)}{g_i^k g_i^{k+1}} &\leq 0. \end{aligned} \tag{19}$$

By hypothesis, $\left(w_i^{k+1} - w_i^k\right) > 0$ and $g_i^k g_i^{k+1} > 0$, and it follows that:

$$R_i'\left(\varepsilon \middle| G_i, Q_i\right) \leq 0 \quad \Leftrightarrow \quad \frac{\left(Q_i^k - Q_i^{k-1}\right)}{\left(Q_i^{k+1} - Q_i^k\right)} \leq \frac{g_i^k\left(1 - G_i^k\right)}{g_i^{k+1}\left(1 - G_i^{k-1}\right)}. \tag{20}$$

Next we argue that in fact the necessary condition for optimality has to be

$$R_i^{'}\left(\varepsilon \middle| G_i, Q_i\right) < 0 \quad \Leftrightarrow \quad \frac{\left(Q_i^k - Q_i^{k-1}\right)}{\left(Q_i^{k+1} - Q_i^k\right)} < \frac{g_i^k\left(1 - G_i^k\right)}{g_i^{k+1}\left(1 - G_i^{k-1}\right)}. \tag{21}$$

The argument is by contradiction and thus suppose that $R_i'(\varepsilon) = 0$ over the entire range of ε. An immediate implication is that the auctioneer would then be indifferent between facing types w_i^k and w_i^{k+1} and all convex combinations represented by (16) and (17). But consider the virtual utilities of these two types, which are given by:

$$\gamma_i^k(\varepsilon) = w_i^k(\varepsilon) - \left(w_i^{k+1}(\varepsilon) - w_i^k(\varepsilon)\right)\frac{1 - G_i^k}{g_i^k}, \tag{22}$$

and

$$\gamma_i^{k+1}(\varepsilon) = w_i^{k+1}(\varepsilon) - \left(w_i^{k+2} - w_i^{k+1}(\varepsilon)\right)\frac{1 - G_i^{k+1}}{g_i^{k+1}}. \tag{23}$$

As ε increases $w_i^k(\varepsilon)$ approaches $w_i^{k+1}(\varepsilon)$ and in consequence, eventually $\gamma_i^{k+1}(\varepsilon) < \gamma_i^k(\varepsilon)$. But by Lemma 2, every information structure with non-monotone virtual utilities is strictly dominated by one with monotone increasing virtual utilities, and hence we have the contradiction. It follows that (21) is a necessary condition for optimality. We argue now that every information structure which is not a monotone partition necessarily fails to satisfy condition (21). Suppose therefore that at least one agent i has an information structure which is not a monotone partition. It follows that there must be two adjacent expected valuations w_i^k and w_i^{k+1}, where we recall that:

$$w_i^k = \frac{\int_0^1 v_i dF_i\left(v_i \left| s_i^k\right.\right)}{\int_0^1 dF_i\left(v_i \left| s_i^k\right.\right)},$$

and an $x \in (0,1)$ such that lower and upper segment of each conditional distribution has strictly positive probability, or:

$$\int_0^x dF_i\left(v_i \middle| s_i^k\right), \int_x^1 dF_i\left(v_i \middle| s_i^k\right), \int_0^x dF_i\left(v_i \middle| s_i^{k+1}\right), \int_x^1 dF_i\left(v_i \middle| s_i^{k+1}\right) > 0.$$

It follows that

$$\frac{\int_x^1 v_i dF_i\left(v_i\left|s_i^k\right.\right)}{\int_x^1 dF_i\left(v_i\left|s_i^k\right.\right)} > \frac{\int_0^x v_i dF_i\left(v_i\left|s_i^{k+1}\right.\right)}{\int_0^x dF_i\left(v_i\left|s_i^{k+1}\right.\right)}. \tag{24}$$

We can represent the expected value w_i^k and w_i^{k+1} as a bundling of the lower and upper segment:

$$w_i^k = \frac{\underline{w}_i^k \underline{g}_i^k + \overline{w}_i^k \overline{g}_i^k}{\underline{g}_i^k + \overline{g}_i^k},\; w_i^{k+1} = \frac{\underline{w}_i^{k+1} \underline{g}_i^{k+1} + \overline{w}_i^{k+1} \overline{g}_i^{k+1}}{\underline{g}_i^{k+1} + \overline{g}_i^{k+1}},$$

with the obvious identification:

$$\underline{w}_i^k = \frac{\int_0^x v_i dF_i\left(v_i\left|s_i^{k+1}\right.\right)}{\int_0^x dF_i\left(v_i\left|s_i^{k+1}\right.\right)},\; \underline{g}_i^k = \int_0^x dF_i\left(v_i\left|s_i^{k+1}\right.\right),$$

and similar for the other components. We can restate (24) as $\overline{w}_i^k > \underline{w}_i^{k+1}$ even though by hypothesis $w_i^k < w_i^{k+1}$. It is this contrast which allows us to come to the conclusion that the optimal information structure must be a monotone partition. For if we were to consider a local modification with the segments $\overline{w}_i^k$ and $\underline{w}_i^{k+1}$, we would get necessary first order conditions of the form

$$\left(\underline{w}_i^{k+1} - \overline{w}_i^k\right) \frac{\left(Q_i^k - Q_i^{k-1}\right) g_i^{k+1}\left(1 - G_i^{k-1}\right) - \left(Q_i^{k+1} - Q_i^k\right) g_i^k\left(1 - G_i^k\right)}{g_i^k g_i^{k+1}} < 0 \tag{25}$$

and in contrast a mixing with w_i^k and w_i^{k+1} leads to a first order condition of the form

$$\left(w_i^{k+1} - w_i^k\right) \frac{\left(Q_i^k - Q_i^{k-1}\right) g_i^{k+1}\left(1 - G_i^{k-1}\right) - \left(Q_i^{k+1} - Q_i^k\right) g_i^k\left(1 - G_i^k\right)}{g_i^k g_i^{k+1}} < 0, \tag{26}$$

but obviously (25) and (26) establish the desired contradiction.

(**1.**) From the previous argument, it follows that if an optimal information structure exists, then it must be a monotone partition. An element P_i^k of a monotone partition $\mathcal{P}_i$ is an interval $P_i^k = [z_i^{k-1}, z_i^k)$ and a point z_i^k is called a boundary point of S_i^k and S_i^{k+1}. We denote by $\mathbf{z}_i$ the vector of all *boundary points* between any two partition elements of bidder i. For every bidder the set of feasible boundary points $\mathbf{z}_i = \{z_i^1, \ldots, z_i^K\}$ is the K dimensional cone defined by the inequalities $0 \leq z_i^1 \leq \ldots \leq z_i^K \leq 1$. The space of feasible boundary points is compact. The expected valuation w_i^k is continuous in the location of the boundary points in the partition. Hence, the virtual utility γ_i^k and, thus, the objective function of the auctioneer are continuous in the location of each boundary point in the partition. By Weierstrass' theorem an optimal partition is guaranteed to exist.■

Proof of Lemma 4. A different set of necessary conditions for an optimal information structure is that the auctioneer does not wish to modify the type probabilities g_i^k of agent i without changing the aggregate winning probabilities of all the other agents. Given a candidate information structure G_i and expected valuations $\left\{w_i^k\right\}_{k=1}^K$ one such local change would result from setting

$$g_i^k(\varepsilon) = g_i^k + \varepsilon \text{ and } g_i^{k+1}(\varepsilon) = g_i^{k+1} - \varepsilon,$$

for $0 < \varepsilon \leq g_i^{k+1}$. It is sufficient to consider a change in the expected value of w_i^k through

$$w_i^k(\varepsilon) = \frac{g_i^k w_i^k + \varepsilon w_i^{k+1}}{g_i^k + \varepsilon},$$

and keep w_i^{k+1} unchanged. For this modification to maintain the winning probabilities of all other agents, it has to be that the aggregate expected probability of winning of agent i remains unchanged for all ε, or

$$g_i^k(\varepsilon) Q_i^k(\varepsilon) + g_i^{k+1}(\varepsilon) Q_i^{k+1} = g_i^k Q_i^k + g_i^{k+1} Q_i^{k+1},$$

where we choose to maintain $Q_i^{k+1}(\varepsilon) = Q_i^{k+1}$ for all ε. As in the earlier argument of Theorem 1, the marginal revenue with respect to changes in ε have to be less than or equal to zero:

$$R_i'(\varepsilon | G_i, Q_i) = g_i^k \left(-g_i^k \left(\left(Q_i^{k+1} - Q_i^k\right) - \left(Q_i^k - Q_i^{k-1}\right)\right) + \left(Q_i^{k+1} - Q_i^{k-1}\right)\varepsilon\right)\left(w_i^{k+1} - w_i^k\right) \frac{1 - G_i^{k-1}}{\left(g_i^k + \varepsilon\right)^3}, \tag{27}$$

and evaluated at $\varepsilon = 0$, we have

$$R_i'(0 | G_i, Q_i) = -\left(\left(Q_i^{k+1} - Q_i^k\right) - \left(Q_i^k - Q_i^{k-1}\right)\right)\left(w_i^{k+1} - w_i^k\right) \frac{1 - G_i^{k-1}}{g_i^k} \leq 0. \tag{28}$$

Finally (27) and (28) jointly imply that $\left(Q_i^{k+1} - Q_i^k\right) > \left(Q_i^k - Q_i^{k-1}\right)$ for all k.■

Proof of Proposition 1. We construct a sequence of mechanisms with finite information structures to establish the desired limiting result. Our construction resembles the well known result that any (Riemann) integrable function is the limit of a sequence of *step* functions. The proof is complicated by the fact that the mechanism along the sequence has to remain incentive compatible. Fix the information structure G and the allocation (q,t). The approximation argument is established by analyzing the associated interim probabilities and transfers, $Q_i(w_i)$ and $T_i(w_i)$, respectively. To prove the Proposition it is sufficient to show that there is a sequence of discrete information structures $\{G_i^n(w_i)\}_{n=1}^{\infty}$ and associated interim incentive and individually rational allocations $\{Q_i^n(w_i), T_i^n(w_i)\}_{n=1}^{\infty}$ for all i such that:

$$\lim_{n\to\infty}\left[\int_0^1 T_i^n(w_i)\,dG_i^n(w_i)\right] = \int_0^1 T_i(w_i)\,dG_i(w_i),\forall i, \tag{29}$$

and

$$\lim_{n\to\infty}\left[\int_0^1 Q_i^n(w_i)\,dG_i^n(w_i)\right] \leq \int_0^1 Q_i(w_i)\,dG_i(w_i),\forall i. \tag{30}$$

Condition (29) guarantees that the expected revenues of information structure $G_i(w_i)$ can be approximated arbitrarily close by a discrete information structure. Condition (30) guarantees that the expected probability by which agent i receives the object is not larger than under the original mechanism and hence that the incentives for the remaining agents in terms of providing the object are not adversely affected by the discrete information structure and modified allocation rule. Provided conditions (29) and (30) can be satisfied, it suffices to give the argument for a particular agent i with independent distributions $F_i(v_i)$ and hence independent information structures $G_i(w_i)$.

By the hypothesis of incentive compatibility, $Q_i(w_i)$ and $T_i(w_i)$ are nondecreasing. By the interim participation constraints, $Q_i(w_i), T_i(w_i) \in [0,1]$. In consequence $Q_i(w_i)$ and $T_i(w_i)$ can have at most a countable number of discontinuities. It also follows from the interim incentive constraints that $Q_i(w_i)$ and $T_i(w_i)$ must have discontinuities on the same set of points. Define the characteristic function of a set E as

$$\mathbb{I}_E(w_i) \triangleq \begin{cases} 1, & \text{if } w_i \in E, \\ 0, & \text{if } w_i \notin E. \end{cases}$$

Based on the original allocation $\{Q_i(w_i), T_i(w_i)\}$ and for every $n = 1,2,3,...,\infty$ and $k = 1,2,....,2^n$, define

$$E_i^{k,n} \triangleq \left\{ w_i \,\middle|\, \frac{k-1}{2^n} \leq T_i(w_i) < \frac{k}{2^n} \right\}. \tag{31}$$

By the monotonicity of $T_i(w_i)$, the set $E_i^{k,n}$ is an interval for every k and n. For every n, the sets $E_i^{k,n}$ generate a monotone partition, denoted by $\mathcal{E}^n$:

$$\mathcal{E}^n \triangleq \left\{E_i^{k,n}\right\}_{k=1}^{2^n}.$$

We identify the associated boundary points of the partition $\mathcal{E}^n$ in the domain of the valuations, $w_i \in [0,1]$, by $\overline{w}_i^{k,n}$, such that:

$$\overline{w}_i^{k-1,n} \leq w_i < \overline{w}_i^{k,n} \quad \Leftrightarrow \quad \frac{k-1}{2^n} \leq T_i(w_i) < \frac{k}{2^n}. \tag{32}$$

For every interval $E_i^{k,n}$, we define the conditional expected valuations $w_i^{k,n}$ by:

$$w_i^{k,n} \triangleq \frac{\int_{E_i^{k,n}} w_i dG_i(w_i)}{\int_{E_i^{k,n}} dG_i(w_i)}. \tag{33}$$

Based on the partition $\mathcal{E}^n$, we define an associated discrete distribution:

$$G_i^n(w_i) \triangleq G_i\left(\overline{w}_i^{k-1,n}\right), \qquad \text{if} \quad w_i^{k-1,n} \leq w_i < w_i^{k,n}. \tag{34}$$

We define a *simple* transfer function $T_i^n(w_i)$, based on the partition $\mathcal{E}^n$, by:

$$T_i^n(w_i) \triangleq T_i\left(\overline{w}_i^{k-1,n}\right) \quad \text{if} \quad \overline{w}_i^{k-1,n} \leq w_i < \overline{w}_i^{k,n}.$$

and a *simple* probability function $Q_i^n(w_i)$ by:

$$Q_i^n(w_i) \triangleq Q_i\left(\overline{w}_i^{k-1,n}\right), \qquad \text{if} \quad \overline{w}_i^{k-1,n} \leq w_i < \overline{w}_i^{k,n}.$$

The simple functions, $Q_i^n(w_i)$ and $T_i^n(w_i)$, converge uniformly to $Q_i(w_i)$ and $T_i(w_i)$, respectively (see Theorem 11.20, Rudin (1964)).

By construction of $T_i^n(w_i)$ and $Q_i^n(w_i)$ it follows that

$$\int T_i^n(w_i)\, dG_i(w_i) \leq \int T_i(w_i)\, dG_i(w_i),$$

as well as

$$\int Q_i^n(w_i)\, dG_i(w_i) \leq \int Q_i(w_i)\, dG_i(w_i),$$

for all n. Moreover, by the dominated convergence theorem (see Theorem 1.6.9, Ash (1972)) it then follows that:

$$\lim_{n\to\infty} \int T_i^n(w_i)\, dG_i(w_i) = \int T_i(w_i)\, dG_i(w_i),$$

as well as

$$\lim_{n\to\infty}\int Q_i^n(w_i)\,dG_i(w_i)=\int Q_i(w_i)\,dG_i(w_i)\,.$$

From the construction of $\{Q_i^n(w_i), T_i^n(w_i)\}$ and $G_i^n(w_i)$ we have that for every n,

$$\int T_i^n(w_i)\,dG_i(w_i)=\int T_i^n(w_i)\,dG_i^n(w_i)\,,$$

as well as

$$\int Q_i^n(w_i)\,dG_i(w_i)=\int Q_i^n(w_i)\,dG_i^n(w_i)\,.$$

The proof is complete if we show that the allocation rule $\{Q_i^n(w_i), T_i^n(w_i)\}$ is interim incentive compatible for the discrete information structure $G_i^n(w_i)$ and the set of expected valuations $w_i^{k,n}\in\left\{w_i^{1,n},....,w_i^{2^n,n}\right\}$ generated by (33). While $\{Q_i^n(w_i), T_i^n(w_i)\}$ will not generally be interim incentive compatible, a straightforward modification, denoted by $\left\{\widehat{Q}_i^n(w_i), \widehat{T}_i^n(w_i)\right\}$ will be. Moreover, the allocation policies will be related as follows:

$$Q_i^n(w_i)=\widehat{Q}_i^n(w_i)\,,\ T_i^n(w_i)\leq\widehat{T}_i^n(w_i)\,, \tag{35}$$

thus only strengthening our claim. The construction proceeds inductively. For a given n, start with $k=1,2$. By construction, either one of the following four cases may occur

$$\begin{array}{ccc} & Q_i^n\left(w_i^{2,n}\right)=Q_i\left(w_i^{2,n}\right) & Q_i^n\left(w_i^{2,n}\right)<Q_i\left(w_i^{2,n}\right)\\ Q_i^n\left(w_i^{1,n}\right)=Q_i\left(w_i^{1,n}\right) & (a) & (b)\\ Q_i^n\left(w_i^{1,n}\right)<Q_i\left(w_i^{1,n}\right) & (c) & (d)\end{array} \tag{36}$$

The local incentive compatibility conditions can be written as

$$w_i^{1,n}\left[Q_i^n\left(w_i^{2,n}\right)-Q_i^n\left(w_i^{1,n}\right)\right]\leq\left[T_i^n\left(w_i^{2,n}\right)-T_i^n\left(w_i^{1,n}\right)\right], \tag{37}$$

and

$$w_i^{2,n}\left[Q_i^n\left(w_i^{2,n}\right)-Q_i^n\left(w_i^{1,n}\right)\right]\geq\left[T_i^n\left(w_i^{2,n}\right)-T_i^n\left(w_i^{1,n}\right)\right]. \tag{38}$$

We now discuss the cases $(a)-(d)$ in (36) sequentially. In case (a), the incentive constraints (37) and (38) are satisfied by the hypothesis of $\{Q_i(w_i), T_i(w_i)\}$ being incentive compatible. In case (b), it follows that incentive compatibility is satisfied for $w_i^{1,n}$ and $w_i^{2,n}$ as it is satisfied by construction for $w_i^{1,n}$ and $\overline{w}_i^{1,n}\leq w_i^{2,n}$. In case (c), the incentive condition for $w_i^{1,n}$, or inequality (37) might be violated as, by hypothesis, (37) is only valid for $\overline{w}_i^{0,n} \leq w_i^{1,n}$. However as $w_i^{1,n} < w_i^{2,n}$, we are guaranteed to find $\widehat{T}_i^n\left(w_i^{2,n}\right) > T_i^n\left(w_i^{2,n}\right)$ which would restore the inequality (37) by means of an equality and leave (38) as a strict inequality. Consider finally case (d). Again, the incentive compatibility condition for $w_i^{2,n}$ has to be satisfied as by hypothesis it is satisfied for $\overline{w}_i^{1,n} < w_i^{2,n}$. It thus follows that only (37) can be violated and that it can again be restored by raising $T_i^n\left(w_i^{2,n}\right)$ to $\widehat{T}_i^n\left(w_i^{2,n}\right)$ so that (37) is restored as an equality. Finally if in either (c) or (d), we raised $T_i^n\left(w_i^{2,n}\right)$ to $\widehat{T}_i^n\left(w_i^{2,n}\right)$, then raise all transfers $T_i^n\left(w_i^{k,n}\right)$ to $\widehat{T}_i^n\left(w_i^{k,n}\right)$ for $k > 2$ by the same amount, namely $\widehat{T}_i^n\left(w_i^{2,n}\right) - T_i^n\left(w_i^{2,n}\right)$. Observe that the participation constraint remains to hold for all valuations $w_i^{k,n}$ even under the higher payments as the lowest type is indifferent and is by construction guaranteed to obtain a nonnegative surplus. Furthermore, the pairwise incentive compatibility conditions for all types k and $k+1$ for $k \geq 2$ remains identical after the uniform raise of the transfers. We can now repeat the modification inductively for all local incentive conditions k and $k+1$, starting with $k = 2$ and $k+1 = 3$. As the discrete information structure has a finite number n of elements, the induction is well-defined. Finally by the single crossing property of the payoff, to verify global incentive compatibility, it is sufficient to verify the local (pairwise adjacent) incentive constraints. Thus we have shown the existence of an incentive compatible mechanism $\left\{\widehat{Q}_i^n(w_i), \widehat{T}_i^n(w_i)\right\}$ for every discrete information structure $G_i^n(w_i)$, which satisfies the properties (35) and this completes the proof.■

Proof of Theorem 2. We first show that an optimal information structure exists in the class of all finite information structures. We then extend the argument to all discrete and finally to all Borel measurable information structures. The proof is by contradiction and relies on the convexity of the conditional winning probabilities.

By Theorem 1, for every finite K, the optimal information structure is a monotone partition for every agent i. We recall from Theorem 1 that an element P_i^k of a monotone partition $\mathcal{P}_i$ is an interval $P_i^k = [z_i^{k-1}, z_i^k)$ and a point z_i^k is a *boundary point* of P_i^k and P_i^{k+1}. We denote by

$$\mathbf{z}_i^K = \left(z_i^{0,K}, z_i^{1,K}, ..., z_i^{K,K}\right)$$

a vector of boundary points of bidder i with $k \in \{0, 1, ..., K\}$. Without loss of generality, we may take the first element in the partition $\left[z_i^{0,K}, z_i^{1,K}\right] = \left[0, z_i^{1,K}\right]$ to be the partition element with nonpositive virtual utility, or $\gamma_i^{1,K} \leq 0$. Conversely, all remaining elements have strictly positive virtual utility $\gamma_i^{k,K} > 0$ for all $k > 1$. If it happens that all elements in the partition have strictly positive virtual utility, then we can simply set $z_i^{1,K} = 0$.

For every finite K, we can take $\mathbf{z}_i^K$ to be an element of the infinite countable product space $[0,1]^\infty$, with the property that $z_i^k = 1$ for all $k > K$. The space $[0,1]^\infty$ is compact in the product topology by Tychonoff's theorem (see Munkres (2000), Theorem 37.3). The infinite countable product space is metrizable in the product topology (see Munkres (2000), Theorem 20.5). The Bolzano-Weierstrass property then states that in every compact metric space every sequence has a convergent subsequence (see Munkres (2000), Theorem 28.2).

Suppose now by way of contradiction that as $K \to \infty$, there is at least on agent i who is assigned an ever increasing number of signals, all (but one) of which have a strictly positive conditional expected probability. Consider the limiting information structure as $K \to \infty$ for this agent i. By the Bolzano-Weierstrass property this limit is well-defined (for a subsequence if necessary). For notational convenience we denote all the limiting values with an upper bar and the limiting partition is given $\overline{\mathbf{z}}_i = \left(\overline{z}_i^0, \overline{z}_i^1, ...\right)$ and correspondingly the limiting conditional probabilities are given by $\overline{\mathbf{Q}}_i = \left(\overline{Q}_i^1, \overline{Q}_i^2, ...\right)$. The limiting partition preserves the monotonicity and the limiting conditional winning probabilities will satisfy weak convexity.

It will be sufficient to look at the limit of the conditional probability of the type of agent i with the lowest strictly positive virtual utility for every finite K, or

$$\lim_{K \to \infty} Q_i^{2,K} = \overline{Q}_i^2.$$

If $\overline{Q}_i^1 > 0$, then we have an immediate contradiction to the hypothesis of an infinite information structure. As the first order conditions for every finite K require that the conditional winning probabilities have to be strictly convex (see Lemma 4), the limiting values have to be weakly convex as well. But as $Q_i^{k,K} \in [0,1]$ and hence $\overline{Q}_i^k \in [0,1]$ as well, $\overline{Q}_i^2 > 0$ implies that only a finite number, and in fact at most $1/\overline{Q}_i^2$ signals can have a positive probability of winning in the limiting information structure.

The case of $\overline{Q}_i^2 = 0$ can be discarded by a similar but slightly more subtle argument. Suppose then that $\overline{Q}_i^2 = 0$. This implies that there must be at least one other agent, say j, who has a positive probability of winning for all his true types in the limiting distribution, or

$$\lim_{K\to\infty} \Pr\left(0 \leq v_j \leq z_j^{1,K}\right) = 0.$$

For else $Q_i^{2,K}$ could not converge to $\overline{Q}_i^2 = 0$ even though by construction $\gamma_i^{2,K} > 0$ for all K. Thus for every valuation $v_j > 0$, it must be that as K becomes large, eventually $v_j \in \left[z_j^{k-1,K}, z_j^{k,K}\right]$ with $k > 1$ and hence

$$\lim_{K\to\infty} z_j^{1,K} = \overline{z}_j^1 = 0.$$

We can now show that limiting probability of the second element in the partition of agent j must be strictly positive and bounded away from zero, or

$$\Pr\left(\overline{z}_j^1 \leq v_j \leq \overline{z}_j^2\right) = \int_{\overline{z}_j^1}^{\overline{z}_j^2} f_j\left(v_j\right) dv_j > 0.$$

The proof is by contradiction and we suppose that

$$\lim_{K\to\infty} \Pr\left(z_j^{1,K} \leq v_j \leq z_j^{2,K}\right) = 0. \qquad (39)$$

By construction, $\gamma_j^{2,K} > 0$ for all K. The virtual utility $\gamma_j^{2,K}$ is given by

$$\gamma_j^{2,K} = w_j^{2,K} - \left(w_j^{3,K} - w_j^{2,K}\right) \frac{1 - G_j^{2,K}}{g_j^{2,K}}.$$

The virtual utility $\gamma_j^{2,K}$ is clearly bounded above by:

$$\gamma_j^{2,K} = w_j^{2,K} - \left(w_j^{3,K} - w_j^{2,K}\right) \frac{1 - G_j^{2,K}}{g_j^{2,K}} < w_j^{2,K} - \left(z_j^{3,K} - w_j^{2,K}\right) \frac{1 - G_j^{2,K}}{g_j^{2,K}}, \qquad (40)$$

as $z_j^{2,K} < w_j^{3,K}$. By way of contradiction, we can now ask what would happen to the value of this upper bound if $z_j^{2,K}$ were to converge to 0 as K grows large:

$$\lim_{z_j^{2,K} \downarrow z_j^{1,K}} \left\{ w_j^{2,K} - \left(z_j^{2,K} - w_j^{2,K}\right) \frac{1 - G_j^{2,K}}{g_j^{2,K}} \right\}. \qquad (41)$$

If $z_j^{2,K} \to z_j^{1,K}$ (and by construction $z_j^{1,K} \to 0$), then we have $\left(z_j^{2,K} - w_j^{2,K}\right) \to 0$ as well as $g_j^{2,K} \to 0$. We therefore have to use l'Hopital's rule to establish the limit behavior of (41). We can write

$$-\left(z_j^{2,K} - w_j^{2,K}\right)\frac{1}{g_j^{2,K}} = -\frac{z_j^{2,K}\int_{z_j^{1,K}}^{z_j^{2,K}} f_j\left(v_j\right)dv_j - \int_{z_j^{1,K}}^{z_j^{2,K}} v_j f_j\left(v_j\right)dv_j}{\left(\int_{z_j^{1,K}}^{z_j^{2,K}} f_j\left(v_j\right)dv_j\right)^2}$$

and after differentiating denominator and numerator separately with respect to $z_j^{2,K}$ to get

$$-\frac{\int_{z_j^{1,K}}^{z_j^{2,K}} f_j\left(v_j\right)dv_j + z_j^{2,K} f_j\left(z_j^{2,K}\right) - z_j^{2,K} f_j\left(z_j^{2,K}\right)}{2\left(\int_{z_j^{1,K}}^{z_j^{2,K}} f_j\left(v_j\right)dv_j\right) f_j\left(z_j^{2,K}\right)},$$

which after elimination of terms is equal to

$$-\frac{1}{2f_j\left(z_j^{2,K}\right)}.$$

As $z_j^{2,K} \to z_j^{1,K}$ and also $z_j^{1,K} \to 0$, we find that

$$\lim_{z_j^{2,K} \downarrow z_j^{1,K}} \left\{w_j^{2,K} - \left(z_j^{2,K} - w_j^{2,K}\right)\frac{1-G_j^{2,K}}{g_j^{2,K}}\right\} = -\frac{1}{2f_j\left(0\right)} < 0.$$

By assumption, the density $f_j\left(v_j\right)$ is positive everywhere and hence an upper bound for the limiting virtual utility $\overline{\gamma}_j^2$ would become strictly negative if the hypothesis (39) were to hold. This delivers the contradiction as by construction $\overline{\gamma}_j^2 \geq 0$. This shows that for agent j the limiting probability of first partition element with positive virtual utility must be strictly positive as $\overline{z}_j^2 > 0$. We denote the limiting probability by:

$$\overline{g}_j^2 = \Pr\left(\overline{z}_j^1 \leq v_j \leq \overline{z}_j^2\right) = \Pr\left(0 \leq v_j \leq \overline{z}_j^2\right) > 0.$$

The limiting conditional winning probability of agent j with his second partition element is now

$$\overline{Q}_j^2 = \prod_{l \neq j} \Pr\left(\overline{\gamma}_l = \overline{\gamma}_l^1\right) > 0.$$

It follows that in the limiting information structure, all types $k > 1$ of all agents except j, and in particular agent i have positive conditional probability of winning of at least:

$$\overline{Q}_l^k \geq \overline{g}_j^2 \prod_{l \neq j} \Pr\left(\overline{\gamma}_l = \overline{\gamma}_j^1\right), \text{ for } k > 1$$

by Corollary 1. But now we can again appeal to the convexity of the conditional winning probabilities to conclude that at most a finite number, in fact no more than

$$\frac{1}{\overline{g}_j^2 \prod_{l \neq j} \Pr\left(\overline{\gamma}_l = \overline{\gamma}_l^1\right)}$$

of types have a positive probability of winning, which contradicts the hypothesis of an information structure with unboundedly many winning types. We have now established that the limiting information structure has a finite number of elements. In the product topology, a function is continuous if it is continuous component by component. As we have only a finite number of components, by the convergence property of the sequence, it follows immediately that the revenue at the limit is equal to limit of the revenues. We can then conclude that the supremum of expected revenues is reached at a finite K, and that the supremum can therefore obtained as the maximal solution of a finite information structure.

Consider next the class of all discrete information structures, finite and infinite. The argument is again by contradiction. Suppose thus that no finite information structure obtains the supremum of revenues. By Proposition 1 there must exist a strictly increasing sequence of K_n, with $K_n < K_{n+1}$, such that $R_{K_n} < R_{K_{n+1}}$ and such that $\lim_{n\to\infty} R_{K_n}$ attains the supremum. But by the first part of this theorem, there exists $\widehat{K} < \infty$, such that for all $K > \widehat{K}$, $R_K < R_{\widehat{K}}$. It then follows that the supremum is reached at a finite K, and that the supremum can be therefore obtained as the maximal solution of a finite information structure. Finally, the argument for all Borel measurable information structures is identical to the previous one, simply by extending the argument from all discrete to all measurable information structures.■

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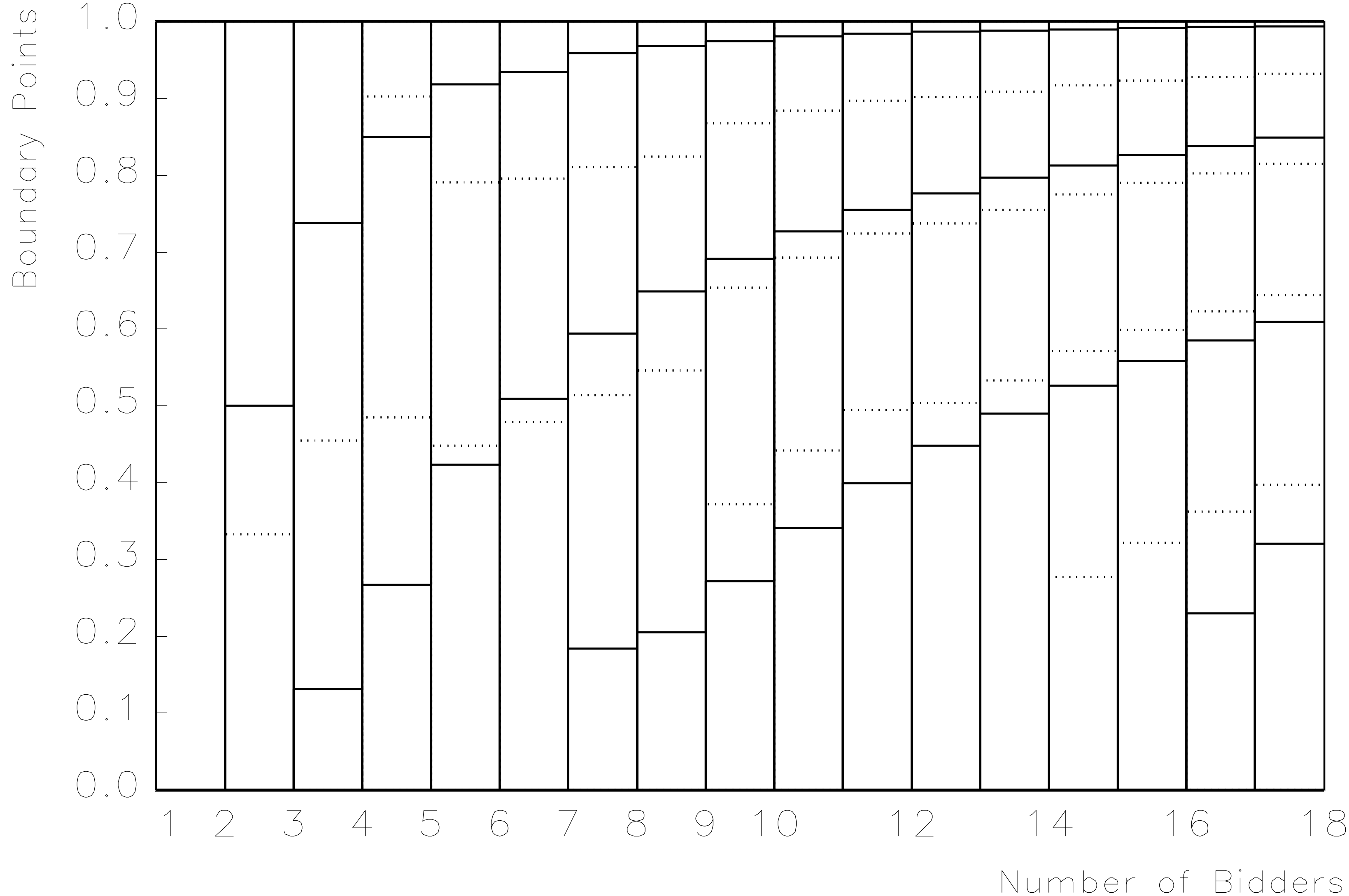
Figure 1: Optimal Partitions



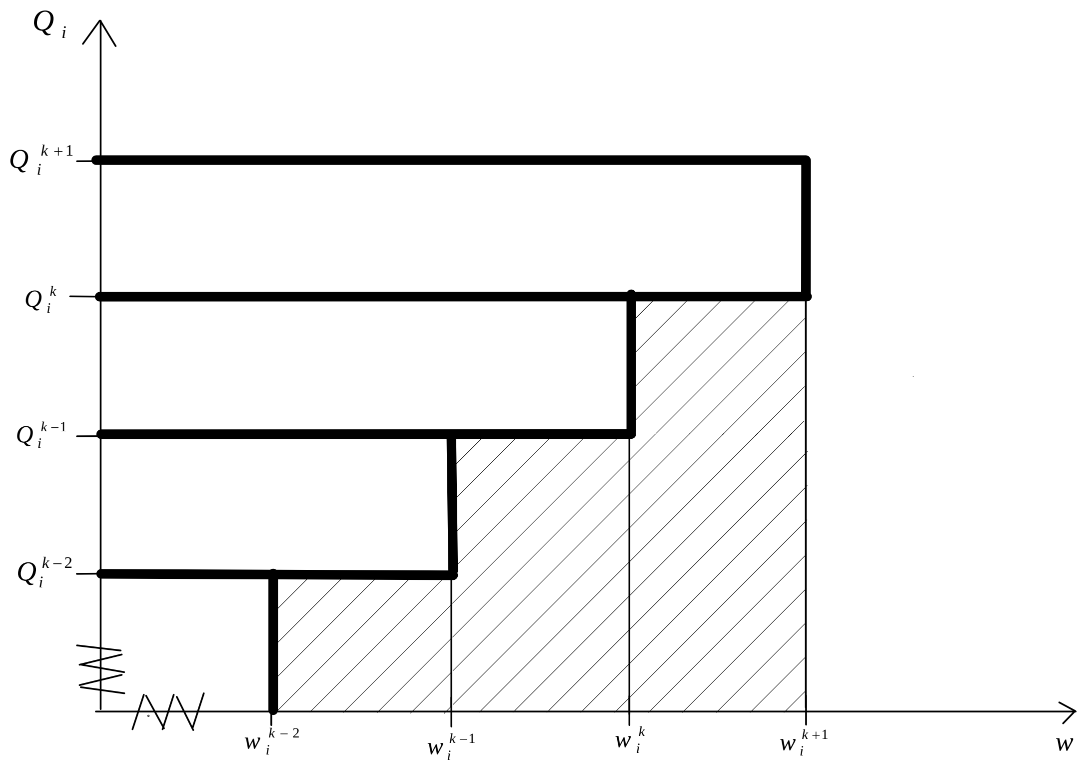


Figure 2a: Surplus Sharing

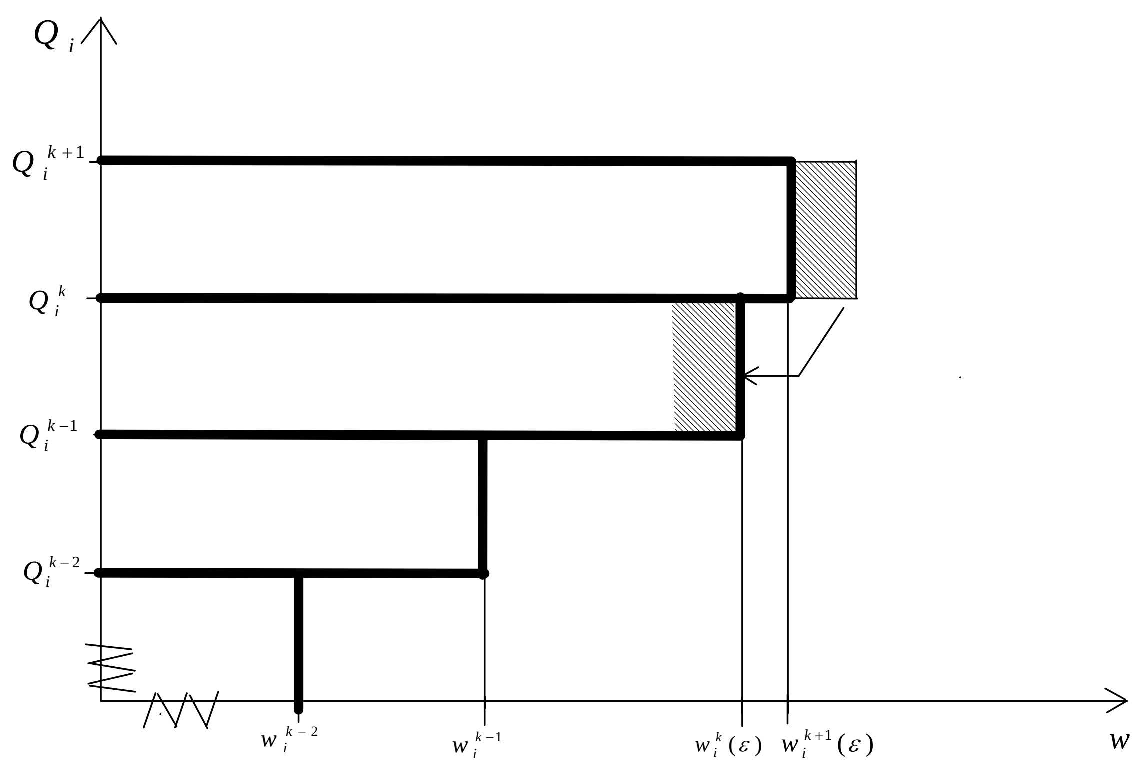


Figure 2b: Mixing of adjacent types by ε