Ec485 Lecture 2, LT2023

1 REVIEW: Random Effect "vs." Fixed Effects

Common misconception: the approaches are frequently thought of as *alternative* DGPs. A much more appropriate framework is to think of them as the *same* DGP, but alternative Estimation Approaches

Common DGP with one-factor error-components model as in (1.8) above:

$$y_{it} = x'_{it}\beta + z'_i\gamma + \epsilon_{it} = x'_{it}\beta + z'_i\gamma + \alpha_i + \nu_{it}$$

RE Approaches: in *RED*: [.]+[.]

$$y_{it} = x'_{it}\beta + z'_i\gamma + \epsilon_{it} = [x'_{it}\beta + z'_i\gamma] + [\alpha_i + \nu_{it}]$$

FE Approaches in *BLACK*: (.) + (.)

$$y_{it} = x'_{it}\beta + z'_i\gamma + \epsilon_{it} = (x'_{it}\beta + z'_i\gamma + \alpha_i) + (\nu_{it})$$

FE-(BLACK): The four classic regression assumptions A1, A2, A3, A4 take the form:

$A1$	no perfect multicollinearity among the regressors X and Z	$rank(X, Z) = k_x + k_z$
$A2$	linear additive model	$y = X\beta + Z\gamma + \epsilon$
$A3$	regressor exogeneity	X and Z exogenous w.r.t. ϵ
$A4$	$VCov(error\|regressors)$	$VCov(\epsilon\|X, Z)$

RE-[RED]: Now the four classic regression assumptions A1, A2, A3, A4 take the form: (D is the full set of N

variable intercepts dummies, one for each individual)

$A1$	no perfect multicollinearity among the regressors X and D	$rank(X, D) = k_x + k_z + N$ $NB:$ Z is dropped since perfectly collinear with D
$A2$	linear additive model	$y = X\beta + Z\gamma + \epsilon = X\beta + D\alpha + \nu$
$A3$	regressor exogeneity	X and D exogenous w.r.t. ν $(no$ Z $regressors)$
$A4$	$VCov(error \mid regressors)$	$VCov(\nu \mid X, D)$

1.1 *FE-TYPE estimators: the α_i's are eliminated through suitable transformation or conditioned upon or estimated through sufficient statistics

Key conclusion: Parameters estimated (either explicitly or implicitly): β (k_x) and $a_1, \cdots, a_N$ (N), σ_ν^2 (1)

1.1.1 FE1: FD

***Apply OLS on FD model:

$$\begin{aligned} \Delta y_{it} &= \Delta x_{it}'\beta + \Delta z_i'\gamma + \Delta\alpha_i + \Delta\nu_{it} \\ &= \Delta x_{it}'\beta + 0 + 0 + \Delta\nu_{it} \end{aligned}$$

NB1: No estimates of γ are possible by the approach since Z has dropped out.

NB2: $\Delta\nu_{it}$ is a non-invertible MA(1) process, with known parameter -1. Hence OLS will not be BLUE and we will need to calculate Robust SEs/VCovs

1.1.2 FE2: Quasi-differencing/Within

***Apply OLS on Quasi-Differenced model:

$$\begin{aligned} Qy &= QX\beta + QZ\gamma + Q\alpha + Q\nu \\ &= QX\beta + 0 + 0 + Q\nu \\ &= QX\beta + Q\nu \end{aligned}$$

where Qy has typical element

$$\{Qy\}_{it} = y_{it} - \bar{y}_{i\cdot} \equiv y_{it} - \frac{1}{T_i}\sum_{t=1}^{T_i} y_{it}$$

Consequently, the Q transformation eliminates all time-invariant terms — in particular α and Z.

NB1: No estimates of γ are possible by the approach since Z has dropped out.

NB2: The transformation Q is idempotent (and symmetric, hence a projection matrix). Therefore, the

$$VCov(\nu|X) = Q\sigma_\nu^2 I_{NT} Q' = \sigma_\nu^2 Q \neq \sigma_\nu^2 I_S$$

which is *singular* (it has deficient rank). Recall that $S = \sum_i T_i$ (which simplifies to NT for a balanced PDS). Therefore its generalized inverse will be *itself* and so the GLS estimator to take into account the non-spherical distribution of ν will be *identical* to plain OLS! To see this formally:

$$\begin{aligned} \text{plain OLS} \quad &: \quad \hat{\beta}_{FE2} = \hat{\beta}_W = ((QX)'(QX))^{-1}(QX)'(Qy) \\ \text{GLS} \quad &: \quad \left((QX)'\,(VCov(\nu|X))^{geninv}\,(QX)\right)^{-1}(QX)'\,(VCov(\nu|X))^{geninv}\,(Qy) \\ &= \quad ((QX)'Q(QX))^{-1}(QX)'Q(Qy) = \hat{\beta}_{FE2} = \hat{\beta}_W \end{aligned}$$

NB3: The FE2 model is *numerically* *identical* to the Variable Intercepts OLS model:

$$y = X\beta + D\alpha + \nu$$

because by the Frisch-Waugh-Lovell theorem, linear regression partitioning gives that:

$$\begin{aligned}
\hat{\beta}_{VIols} &= ((M_D X)'(M_D X))^{-1}(M_D X)'(M_D y) : M_D \equiv I_{NT} - D(D'D)^{-1}D' = Q \\
&= ((QX)'(QX))^{-1}(QX)'(Qy) = \hat{\beta}_{FE2} = \hat{\beta}_W \\
\{\hat{\alpha}_{VIols}\}_i &= \bar{y}_{i\cdot} - \bar{x}'_{i\cdot}\hat{\beta}_{FE2}
\end{aligned}$$

1.2 *RE-TYPE estimators:

Key fact: Parameters estimated: β (k_x), γ (k_z), σ_α^2 (1), and σ_ν^2 (1)

Consider model

$$y = [X\beta + Z\gamma] + [\alpha + \nu] = [X\beta + Z\gamma] + [\epsilon] \equiv W\theta + \epsilon$$

RE1: pooled OLS

$$\hat{\theta}_{RE1} = \begin{pmatrix} \hat{\beta}_{RE1} \\ \hat{\gamma}_{RE1} \end{pmatrix} = (W'W)^{-1}W'y$$

NB: This will *not* be BLUE and its *Robust* SEs/VCov must be calculated to allow for the Clustering exhibited by the *block-diagonal* $VCov(\epsilon|X, Z) \equiv \sigma_\epsilon^2\Omega$.

RE2: "the RE"-GLS estimator

$$\begin{aligned}
\hat{\theta}_{RE2} &= \hat{\theta}_{REgls} = \begin{pmatrix} \hat{\beta}_{REgls} \\ \hat{\gamma}_{REgls} \end{pmatrix} \\
&= (W'\Omega^{-1}W)^{-1}W'\Omega^{-1}y \\
&= ([W'\Omega^{-1/2}][\Omega^{-1/2\prime}W])^{-1}[W'\Omega^{-1/2}][\Omega^{-1/2\prime}y] \\
&= ([\Omega^{-1/2\prime}W]'[\Omega^{-1/2\prime}W])^{-1}[\Omega^{-1/2\prime}W][\Omega^{-1/2\prime}y]
\end{aligned}$$

NB1: This estimator will be BLUE and will have the correct SEs/VCov.

NB2: In 1972, Fuller and Battese showed that calculating Ω^{-1}, which is computationally burdensome, can be avoided. Instead, the rotation $\Omega^{-1/2\prime}$ yields the equivalent very straightforward expressions:

$$\begin{aligned}
\Omega^{-1/2\prime} y &= \{y_{it} - \lambda_i \bar{y}_{i\cdot}\} \\
\Omega^{-1/2\prime} X &= \{x_{it} - \lambda_i \bar{x}_{i\cdot}\} \\
\Omega^{-1/2\prime} Z &= \{(1 - \lambda_i) z_i\} \\
\text{where } \lambda_i &= 1 - \sqrt{\frac{\sigma_\nu^2}{\sigma_\nu^2 + T_i \sigma_\alpha^2}}
\end{aligned}$$

Hence the RE2-GLS estimator can be obtained by applying plain OLS on the $\Omega^{-1/2\prime}$-transformed variables.

2 Some Key Issues and Extensions — Static Models:

(Issue 1) RE methods more efficient in general, but *inconsistent* if A3 violated

(Issue 2) FE methods less efficient in general, but consistent even if Xs endogenous w.r.t. alpha_i effects — since they are now part of regressors, which are allowed to be correlated between themselves.

(Issue 3) FE methods cannot estimate gammas in general, since all time-invariant terms are eliminated/conditioned upon.

(Issue 4) Wu-Hausman Specification Tests – RE and FE compared, Rao-Blackwell theorem useful

(Issue 5) FE-type and RE-type methods pose distinct challenges to generalize to Observable Dynamics in PDS models.

(Issue 6) FE-type methods are harder/impossible to generalize to Nonlinear PDS models.

(Issue 7) FE-type methods are less robust/more likely to be seriously inconsistent in the presence of Regressors with Measurement Errors.

2.1 Extensions and Improvements

(4) Make RE more robust to endogeneity — the "Modified RE" estimator. Chamberlain/Mundlak/Hajivassiliou
See URL: <https://eprints.lse.ac.uk/102843/> Section 2
To summarize:

$$\begin{aligned} y_{it} &= x'_{it}\beta + z'_i\gamma + \alpha_i + \nu_{it} \\ &= x'_{it}\beta + z'_i\gamma + \nu_{it} + \alpha^*_i + \bar{x}'_{i\cdot}\xi + z'_i\zeta \\ &= x'_{it}\beta + \bar{x}'_{i\cdot}\xi + z'_i(\gamma + \zeta) + \alpha^*_i + \nu_{it} \end{aligned}$$

by using the following arguments: the key issue is that X and Z are potentially endogenous w.r.t. α_i, which means that any RE-type estimator will be *inconsistent* in that case for β and γ. We formulate that as:

$$\begin{aligned} 0 &\neq E(\alpha_i|X,Z) = g(X,Z) = \\ assumption\ 1 &: = \text{linear function of } X \text{ and } Z \\ assumption\ 2 &: = \text{time-invariant function} \\ &= \bar{x}'_{i\cdot}\xi + z'_i\zeta \end{aligned}$$

Thus, we define:

$$\alpha^*_i \equiv \alpha_i - E(\alpha_i|X,Z) = \alpha_i - \bar{x}'_{i\cdot}\xi + z'_i\zeta$$

Therefore, the redefined regression equation:

$$y_{it} = x'_{it}\beta + \bar{x}'_{i\cdot}\xi + z'_i(\gamma + \zeta) + \alpha^*_i + \nu_{it}$$

is well-specified and does not suffer from regressor-endogeneity w.r.t. α^*_i. Hence, RE-type estimators applied to it will be consistent (and possibly efficient).

(5) Make FE able to estimate gammas also — the "Modified FE" estimator. FE+IVE. Hausman-Taylor 1981 approach

$$
\begin{aligned}
y_{it} &= x_{it}'\beta + z_i'\gamma + \alpha_i + \nu_{it} \\
&= (x_{it}^{Good}|x_{it}^{Bad})' \begin{pmatrix} \beta^{Good} \\ \beta^{Bad} \end{pmatrix} + (z_i^{Good}|z_i^{Bad})' \begin{pmatrix} \gamma^{Good} \\ \gamma^{Bad} \end{pmatrix} + \alpha_i + \nu_{it}
\end{aligned}
$$

The regressor dimensionalities are $k_x^G, k_x^B, k_z^G, k_z^B$ respectively, with $k_x = k_x^G + k_x^B$ and $k_z = k_z^G + k_z^B$.

The following two steps achieve FE-type of estimation that produce also consistent γ estimates:

Step 1: Obtain $\hat{\beta}_{FE2} = \hat{\beta}_W$ using the Q-transformed data $Qy = \{y_{it} - \bar{y}_{i\cdot})$ etc. This will be *consistent* for both β^{Good} and β^{Bad} since the α_i has been eliminated from the equation.

Step 2: Define:

$$
\begin{aligned}
d_i &= \bar{y}_{i\cdot} - \bar{x}_{i\cdot}'\beta = z_i^{Good\prime}\gamma^{Good} + z_i^{Bad\prime}\gamma^{Bad} + \alpha_i + \bar{\nu}_{i\cdot} \\
\hat{d}_i &= \bar{y}_{i\cdot} - \bar{x}_{i\cdot}'\hat{\beta}_{FE2} = z_i^{Good\prime}\gamma^{Good} + z_i^{Bad\prime}\gamma^{Bad} + \alpha_i + \bar{\nu}_{i\cdot} - \bar{x}_{i\cdot}'(\hat{\beta}_{FE2} - \beta)
\end{aligned}
$$

Regressing $\hat{d}_i$ on z_i^{Good} and z_i^{Bad} by OLS would be *inconsistent* because z_i^{Bad} are endogenous regressors w.r.t. α_i. The Hausman-Taylor solution is to use Instrumental Variables estimator using X^{Good} to instrument for Z^{Bad}, which are *valid* (uncorrelated from the errors) and *relevant* (correlated with Z^{Bad}) instruments. The necessary condition for this is that:

$$\text{Number of } X^{Good} \geq \text{Number of } Z^{Bad}$$

[Note: the presence of the estimation error term $(\hat{\beta}_{FE2} - \beta)$ affects only the second-order (VCov(.)) properties of the estimators, because it converges to 0 as $N \to \infty$.)