Dynare & Perturbation continued

Wouter J. Den Haan

University of Amsterdam

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Overview

1. Is the solution always accurate?
2. Blanchard-Kahn conditions

Accuracy

1. Accuracy tests
    1. simple informal ones:
        - similar answers with first & second-order and with solution in levels and in logs
        - solution "makes sense"
    2. formal ones
2. Local or global approximation?

Local or global approximation?

- System: the unknown policy function $g(x)$ is *implicitly* defined by
$$\mathsf{E}_t\left[f\left(g\left(x\right)\right)\right]=0$$
for known $f(\cdot)$.
- Pertubation $=$ Taylor series expansion of $g(x)$ around steady state.

Taylor series & convergence to the truth

- Let $g(x)$ be defined on $[a,b]$
- Let $\tilde{g}_n(x)$ be the n^{th}-order Taylor series expansion

$$\tilde{g}_n(x) = g(\bar{x}) + \left.\frac{\partial g(x)}{\partial x}\right|_{x=\bar{x}} (x-\bar{x}) + \cdots + \left.\frac{\partial^n g(x)}{\partial x^n}\right|_{x=\bar{x}} \frac{(x-\bar{x})^n}{n!}$$

  for known $f(\cdot)$.

- $$\lim_{n \longrightarrow \infty} \tilde{g}_n(x) \stackrel{?}{\longrightarrow} g(x)$$

- if x close enough to $\bar{x}$ even true for fixed n
- what if x is not arbitrarily close to $\bar{x}$

Example with simple Taylor expansion

Truth:

$$f(x) = -690.59 + 3202.4x - 5739.45x^2 + 4954.2x^3 - 2053.6x^4 + 327.10x^5$$

defined on $[0.7, 2]$

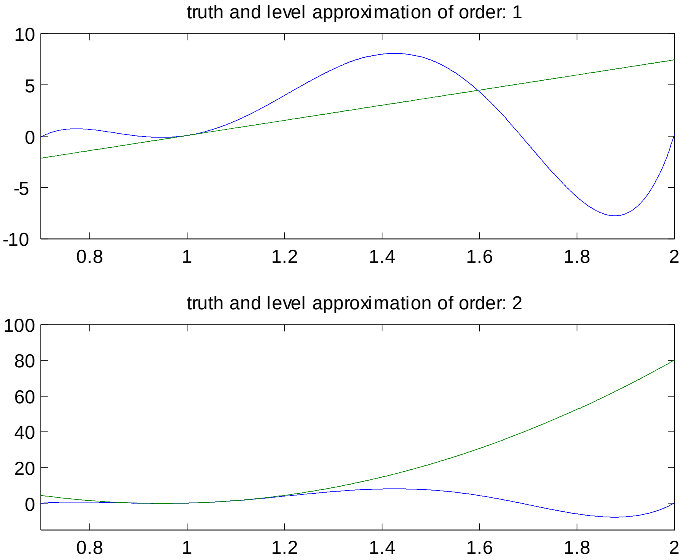


Figure: Level approximations

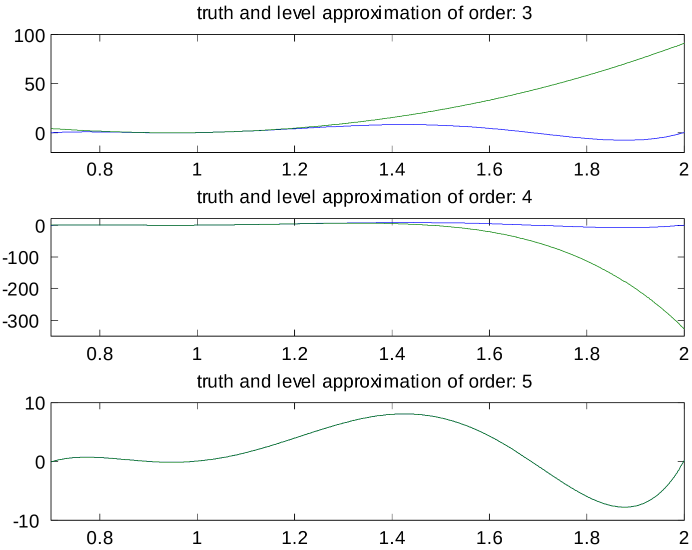


Figure: Level approximations continued

Approximation in log levels

Think of $f(x)$ as a function of $z = \log(x)$. Thus,

$$\begin{aligned} f(x) &= -690.59 + 3202.4\exp(z) - 5739.45\exp(2z) \\ &\quad + 4954.2\exp(3z) - 2053.6\exp(4z) + 327.10\exp(5z). \end{aligned}$$

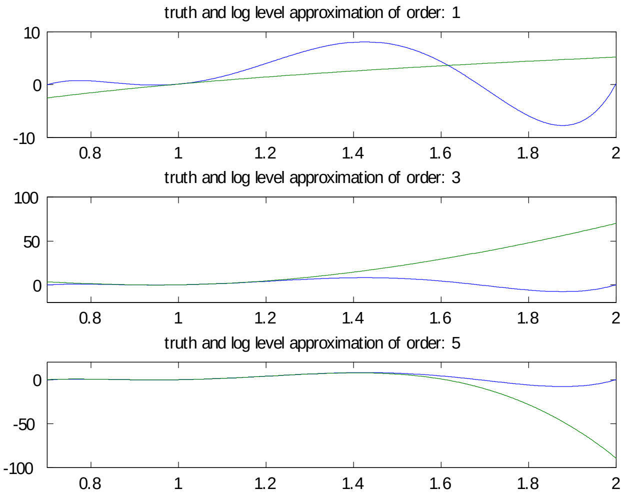


Figure: Log level approximations

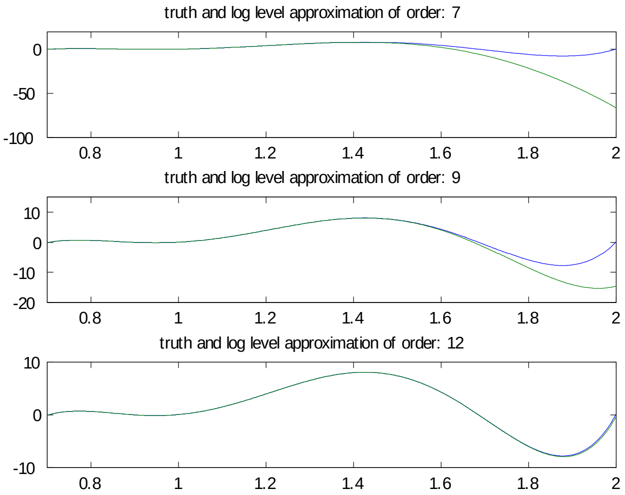
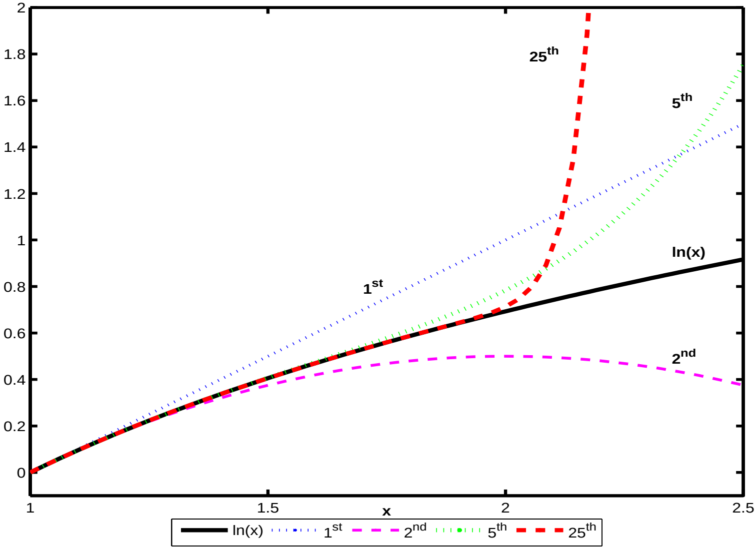


Figure: Log level approximations continued

But global convergence is not guaranteed



$\ln(x)$ and its Taylor series expansions

What to watch out for with perturbation?

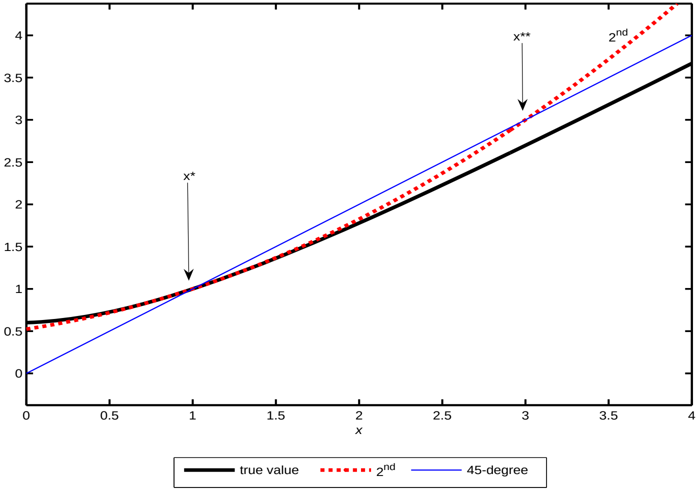
- wrong shape
- explosive behavior

Problems with stability

$$
\begin{aligned}
h(x) &= \alpha_0 + x + \alpha_1 e^{-\alpha_2 x} \\
x_{+1} &= h(x) + \mathsf{shock}_{+1}
\end{aligned}
$$

- Unique globally stable fixed point

Perturbation approximation & stability

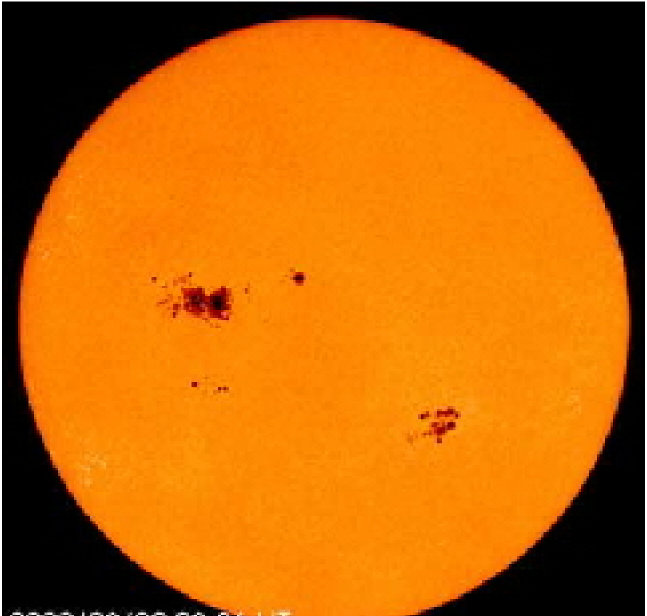


What are the BK conditions about?

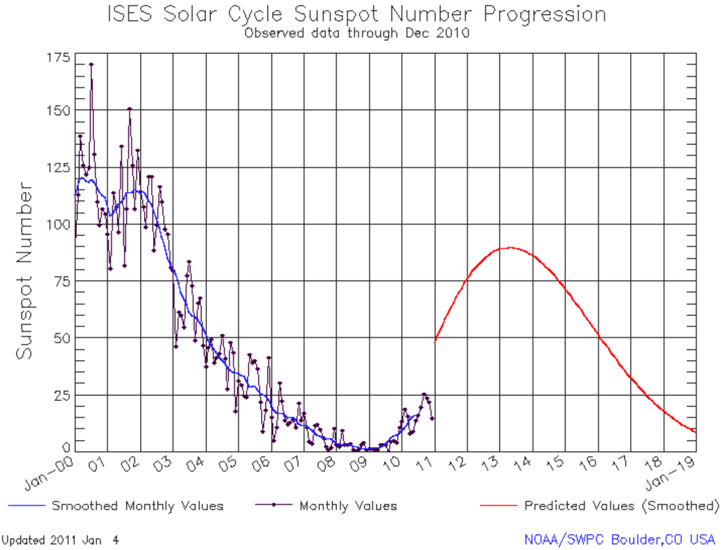
- They say something about *linear* models
- They determine whether
  - a stable solution exists, and
  - if a stable solution exists, whether it is unique

If a solution is *not* unique, then there are many solutions including sun spot solutions

Large sun spots (around 2000 at the peak)



Sun spot cycle (almost at the peak)



Getting started

Model:

$$y_t = \rho y_{t-1}$$

- infinite number of solutions, independent of the value of ρ

Getting started

Model:

$$y_t = \rho y_{t-1}$$
$$y_0 \text{ is given}$$

- unique solution, independent of the value of ρ

Getting started

- Blanchard-Kahn conditions apply to models that add as a requirement that the series do not explode

Model:

$$y_t = \rho y_{t-1}$$

y_t cannot explode

- $\rho > 1$: nique solution, namely $y_t = 0$ for all t
- $\rho < 1$: many solutions
- $\rho = 1$: many solutions
  - be careful with $\rho = 1$, uncertainty matters

State-space representation

$$Ay_{t+1} + By_t = \varepsilon_{t+1}$$

$$\mathsf{E}\left[\varepsilon_{t+1} | I_t\right] = 0$$

y_t is a $n \times 1$ vector

m elements of y_1 are not determined

some elements of ε_{t+1} are not exogenous shocks but prediction errors

Neoclassical growth model and state-space representation

$$\left(\exp(z_t)k_t^{\alpha} + (1-\delta)k_t - k_{t+1}\right)^{-\gamma} =$$
$$\mathsf{E}\left[\left.\begin{array}{c}\beta\left(\exp(z_{t+1})k_{t+1}^{\alpha} + (1-\delta)k_{t+1} - k_{t+2}\right)^{-\gamma} \\ \times\left(\alpha\exp\left(z_{t+1}\right)k_{t+1}^{\alpha-1} + 1 - \delta\right)\end{array}\right| I_t\right]$$

or

$$\left(\exp(z_t)k_t^{\alpha} + (1-\delta)k_t - k_{t+1}\right)^{-\gamma} =$$
$$\beta\left(\exp(z_{t+1})k_{t+1}^{\alpha} + (1-\delta)k_{t+1} - k_{t+2}\right)^{-\gamma}$$
$$\times\left(\alpha\exp\left(z_{t+1}\right)k_{t+1}^{\alpha-1} + 1 - \delta\right)$$
$$+e_{\mathsf{E},t+1}$$

and

$$z_{t+1} = \rho z_t + e_{z,t+1}$$

Neoclassical growth model and state-space representation

Linearized model:

$$k_{t+2} = a_1 k_{t+1} + a_2 k_t + a_3 z_{t+1} + a_4 z_t + e_{\mathrm{E},t+1}$$

$$z_{t+1} = \rho z_t + e_{z,t+1}$$

$$k_1 \text{ is given}$$

- k_t is beginning-of-period capital
  - $\Longrightarrow k_{t+2}$ is chosen in $t+1$

Neoclassical growth model and state-space representation

$$\begin{bmatrix} 1 & 0 & a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_{t+2} \\ k_{t+1} \\ z_{t+1} \end{bmatrix} + \begin{bmatrix} a_1 & a_2 & a_4 \\ -1 & 0 & 0 \\ 0 & 0 & -\rho \end{bmatrix} \begin{bmatrix} k_{t+1} \\ k_t \\ z_t \end{bmatrix} = \begin{bmatrix} e_{\mathsf{E},t+1} \\ 0 \\ e_{z,t+1} \end{bmatrix}$$

Dynamics of the state-space system

$$\begin{aligned} y_{t+1} &= -A^{-1}By_t + A^{-1}\varepsilon_{t+1} \\ &= Dy_t + A^{-1}\varepsilon_{t+1} \end{aligned}$$

Thus

$$y_{t+1} = D^t y_1 + \sum_{l=1}^{t+1} D^{t+1-l}A^{-1}\varepsilon_l$$

Jordan matrix decomposition

$$D = P\Lambda P^{-1}$$

- Λ is a diagonal matrix with the eigen values of D
- without loss of generality assume that $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$

Let

$$P^{-1} = \begin{bmatrix} \tilde{p}_1 \\ \vdots \\ \tilde{p}_n \end{bmatrix}$$

where $\tilde{p}_i$ is a $(1 \times n)$ vector

Dynamics of the state-space system

$$
\begin{aligned}
y_{t+1} &= D^t y_1 + \sum_{l=1}^{t+1} D^{t+1-l} A^{-1} \varepsilon_l \\
&= P \Lambda^t P^{-1} y_1 + \sum_{l=1}^{t+1} P \Lambda^{t+1-l} P^{-1} A^{-1} \varepsilon_l
\end{aligned}
$$

Dynamics of the state-space system

multiplying dynamic state-space system with P^{-1} gives

$$P^{-1}y_{t+1} = \Lambda^t P^{-1} y_1 + \sum_{l=1}^{t+1} \Lambda^{t+1-l} P^{-1} A^{-1} \varepsilon_l$$

or

$$\tilde{p}_i y_{t+1} = \lambda_i^t \tilde{p}_i y_1 + \sum_{l=1}^{t+1} \lambda_i^{t+1-l} \tilde{p}_i A^{-1} \varepsilon_l$$

recall that y_t is $n \times 1$ and $\tilde{p}_i$ is $1 \times n$. Thus, $\tilde{p}_i y_t$ is a scalar

Model

1. $\tilde{p}_i y_{t+1} = \lambda_i^t \tilde{p}_i y_1 + \sum_{l=1}^{t+1} \lambda_i^{t+1-l} \tilde{p}_i A^{-1} \varepsilon_l$
2. $\mathsf{E}[\varepsilon_{t+1} | I_t] = 0$
3. m elements of y_1 are not determined
4. y_t cannot explode

Reasons for multiplicity

1. There are free elements in y_1
2. The only constraint on $e_{\mathsf{E},t+1}$ is that it is a prediction error.
   - This leaves lots of freedom

Eigen values and multiplicity

- Suppose that $\lambda_1 > 1$
- To avoid explosive behavior it *must* be the case that

1. $\tilde{p}_1 y_1 = 0 \quad$ and
2. $\tilde{p}_1 A^{-1} \varepsilon_l = 0 \quad \forall l$

How to think about #1?

$$\tilde{p}_1 y_1 = 0$$

- Simply an additional equation to pin down some of the free elements
- Much better: This is the policy rule in the first period

How to think about #1?

$$\tilde{p}_1 y_1 = 0$$

Neoclassical growth model:

- $y_1 = [k_2, k_1, z_1]^T$
- $\lambda_1 > 1$, $\lambda_2 < 1$, $\lambda_3 = \rho < 1$
- $\tilde{p}_1 y_1$ pins down k_2 as a function of k_1 and z_1
    - this is the policy function in the first period

How to think about #2?

$$\tilde{p}_1 A^{-1} \varepsilon_l = 0 \ \ \forall l$$

- This pins down $e_{\mathsf{E},t}$ as a function of $\varepsilon_{z,t}$
- That is, the prediction error must be a function of the structural shock, $\varepsilon_{z,t}$, and cannot be a function of other shocks,
    - i.e., there are no sunspots

How to think about #2?

$$\tilde{p}_1 A^{-1} \varepsilon_l = 0 \quad \forall l$$

Neoclassical growth model:

- $\tilde{p}_1 A^{-1} \varepsilon_t$ says that the prediction error $e_{\mathrm{E},t}$ of period t is a fixed function of the innovation in period t of the exogenous process, $e_{z,t}$

How to think about #1 combined with #2?

$$\tilde{p}_1 y_t = 0 \quad \forall t$$

- Without sun spots
  - i.e. with $\tilde{p}_1 A^{-1} \varepsilon_t = 0 \ \ \forall t$
- k_{t+1} is pinned down by k_t and z_t in every period.

Blanchard-Kahn conditions

- Uniqueness: For every free element in y_1, you need one eigenvalue that is larger than one
- Multiplicity: Not enough eigenvalues larger than one
- No stable solution: Too many eigenvalues larger than one