Dynare Summer School - Accuracy Tests

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Accuracy tests

- Several models can be solved accurately with simple algorithms
- Several cannot
- Some models cannot be solved accurately with complex algorithms
- How do you know whether your solution is accurate?

How to check for accuracy

1. Informal accuracy tests
2. Formal accuracy tests

Informal accuracy tests

These are possibly more important than formal ones

1. Play with your model/algorithm
    1. Understand properties of the model
    2. Change parameter values and understand how model properties change
    3. Open up the black box
2. Solve your model in a different way
    1. Linear instead of log-linear
    2. Use model equations to substitute out variables

Formal accuracy tests

1. DenHaan-Marcet accuracy test
   - simple
2. Euler equation errors
   - better than DHM (in my opinion)
   - require numerical integration (but this is not that difficult and very good to know anyway)
3. Dynamic Euler equation errors
4. Welfare measures (be careful)

Framework for accuracy tests

Framework:

$$\mathrm{E}\left[f\left(x_{t-1}, x_t, y_t, y_{t+1}\right) | I_t\right] = 0$$

where I is the information set of information available in the current period.

This implies

$$\mathrm{E}\left[f\left(x_{t-1}, x_t, y_t, y_{t+1}\right) h(z_t)' | I_t\right] = 0$$

where z is an element of I and $h(z)$ is any measurable vector-valued $n_h \times 1$ function.

Idea behind accuracy tests:

- DHM: $f\left(x_{t-1}, x_t, y_t, y_{t+1}\right)$ should be a forecast error, that is, not correlated with anything in the information set
- Euler equation error: expected value of $f(\cdot)$ should be zero at many points in the state space

Forecast error in standard growth model

$$c_t^{-\gamma} - \beta c_{t+1}^{-\gamma}(\alpha \exp(\theta_{t+1}) k_t^{\alpha-1} + 1 - \delta)$$

DenHaan-Marcet Accuracy test

$$\mathrm{E}\left[f\left(x_{t-1}, x_t, y_t, y_{t+1}\right) h(z_t)' | I_t\right] = 0$$

This implies that the unconditional expectation should be equal to zero as well:

$$\mathrm{E}\left[f\left(x_{t-1}, x_t, y_t, y_{t+1}\right) h(z_t)'\right] = 0$$

Using simulated data

$$\frac{\sum_{t=1}^{T} f\left(x_{t-1}, x_t, y_t, y_{t+1}\right) h(z_t)'}{T} \approx 0$$

This you can test

DenHaan-Marcet Accuracy test

1. Simulate data of length T (if you are not sure your initial condition is sensible discard a couple hundred observations). Reasonable values are $T = 3,500$ and discard 500.
2. Calculate

$$J_T = TM_T' W_T^{-1} M_T$$

$$M_T = \frac{\sum_{t=1}^{T} h(z_t) f(x_{t-1}, x_t, y_t, y_{t+1})}{T}$$

$$W_T = \frac{\sum_{t=1}^{T} f(x_{t-1}, x_t, y_t, y_{t+1}) h(z_t)' h(z_t) f(x_{t-1}, x_t, y_t, y_{t+1})}{T}$$

This has a χ^2 distribution with n_h degrees of freedom
If $h(z_t)$ is a scalar, then

$$J_T = \left(\frac{M_T}{\sqrt{W_T / T}} \right)^2$$

Implementation of DenHaan-Marcet statistic

1. Do the DHM statistic N times
2. Check the fraction of times the statistic is in the lower and upper 5% range; inaccurate solutions are typically blown away (because of having too many realizations in the upper critical region)
3. Personally, I prefer to do the test multiple times for scalar $h(z_t)$ because this provides more information. In fact, using $h(z_t) = 1$ can already be quite informative

Limits of DenHaan-Marcet statistic

1. Even accurate solutions are rejected more often than 5% for high enough T; thus the higher the value of T for which you get good results the better
2. Results are random so inaccurate solutions could get through by sheer chance
3. The opposite of #2 turns out to be a bigger problem in practice: DHM is difficult to pass in the sense that solutions that in many aspects are close to the true or an extremely accurate solution can fail the DHM statistic miserably

Simulating yourself after you solved model with Dynare

1. Add the following command at the end of the file

   save dynarerocks decision

   The matrix "decision" contains the coefficients of the policy rules *exactly the way they appear on your screen*. This command saves them to the file dynarerocks.mat
2. Reload the matrix "decision" into your program again with the command

   load dynarerocks
3. To make programming easy create labels for columns of "decision"

- iiicons = 4 (if consumption is the fourth variable)
- iiihours = 6 (if hours is the sixth variable)

Implement DHM statistic with Dynare/Matlab

1. Simulate exogenous random shocks
2. Choose z_t and $h(z_t)$
3. Simulate required data as described above
4. Calculate J_T statistic
5. Calculate critical values
    - crit5=chi2inv(0.05,dgf) where dgf=n_h
    - crit95=chi2inv(0.95,dgf) where dgf=n_h
6. Repeat a couple times

Euler equation errors

- True solution satisfies

$$E\left[f\left(x_{t-1}, x_t, y_t, y_{t+1}\right)\middle| I_t\right] = 0$$

  for *all* points in the state space
- This can be checked for *any* numerical solution (including perturbation solutions) at many points in the state space

How to deal with integration?

- Easy if shocks have discrete support. Note that Dynare allows for shocks to be discrete
- Numerical integration (this must of course be done very accurately)

Standard growth model with discrete innovations

$$\max_{\{c_t,k_t\}_{t=1}^{\infty}} \mathrm{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma}-1}{1-\gamma}$$

$$\text{s.t.} \quad c_t + k_t = \exp(\theta_t) k_{t-1}^{\alpha} + (1-\delta) k_t \tag{1}$$

$$\theta_t = \rho \theta_{t-1} + \sigma e_t, \tag{2}$$

$$e_t = \begin{cases} +1 \text{ with probability } 1/2 \\ \text{-1 with probability } 1/2 \end{cases}$$

Standard growth model with discrete innovations

Basic idea

1. Construct fine grid with values for k_{-1} and θ_{-1} and two values for ε.
2. Calculate Euler-equation errors (using bold symbols for policy functions)

$$
\begin{aligned}
&-\mathbf{c}(k_{-1},\theta_{-1},\varepsilon)^{-\gamma} \\
&+0.5 * \beta \mathbf{c}(\mathbf{k},\theta,\sigma)^{-\gamma}(\alpha \exp(\rho\theta+\sigma)\mathbf{k}^{\alpha-1}+1-\delta) \\
&+0.5 * \beta \mathbf{c}(\mathbf{k},\theta,-\sigma)^{-\gamma}(\alpha \exp(\rho\theta-\sigma)\mathbf{k}^{\alpha-1}+1-\delta)
\end{aligned}
$$

$\mathbf{k} = \mathbf{k}(k_{-1},\theta_{-1},\varepsilon)$.

3. Problem is that numbers are difficult to interpret.

Standard growth model with discrete innovations

Interpretable Euler equation errors

1. At each grid point calculate
   - $\mathbf{c}(k_{-1}, \theta_{-1}, \varepsilon)$ and
   - the RHS of the Euler equation

$$g = \begin{array}{l} +0.5 * \beta \mathbf{c}(\mathbf{k}, \theta, +\sigma)^{-\gamma}(\alpha \exp(\rho\theta + \sigma)\mathbf{k}^{\alpha-1} + 1 - \delta) \\ +0.5 * \beta \mathbf{c}(\mathbf{k}, \theta, -\sigma)^{-\gamma}(\alpha \exp(\rho\theta - \sigma)\mathbf{k}^{\alpha-1} + 1 - \delta) \end{array}$$

   and use this to calculate the value of consumption *implied* by g, that is,$\mathbf{c}_{imp}(k_{-1}, \theta_{-1}, \varepsilon) = g^{-1/\gamma}$
   - calculate the percentage difference between c and c_{imp}
2. Calculate maximum and average of the absolute percentage errors
3. Investigate
   - Pattern (e.g., are errors always of the same sign)
   - How likely are the nodes where largest errors occur?
   - Are percentage errors reasonable at nodes where largest errors occur? For example, if consumption is very small than small basically irrelevant errors may show up as large percentage errors

Euler equation errors - Pros & Cons

1. This is probably best that is currently available
2. However, it only tests for one-period ahead forecast errors and thus ignores the possibility of accumulation of small errors
    - DHM statistic could pick those up
    - Dynamic Euler equation error could pick those up too

Dynamic Euler equation errors

1. Generate time series for θ_t and choose k_0
2. Use your numerical solution to generate time series for c_t and k_t
3. Generate an alternative time series
    - use your your numerical solution to calculate conditional expectation
    - calculate consumption from the Euler equation
    - calculate capital from budget constraint

Step 2 detailed

1. Generate time series for θ_t and set $k_{imp,0} = k_0$
2. Calculate g_t (conditional expectation) as explained above. Note you use your numerical solution for the function inside the integral
3. Calculate $c_{imp,t} = g^{-1/\gamma}$
4. Calculate $k_{imp,t} = \theta_t k_{imp,t-1}^{\alpha} + (1-\delta) k_{imp,t-1} - c_{imp,t}$

Household side

$$\max_{\{C_t,K_t\}_{t=1}^{\infty}} \mathrm{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{C_t^{1-\gamma}-1}{1-\gamma}$$

$$\text{s.t.} \quad C_t + K_t = W_t N_{t-1} + R_t K_{t-1} + (1-\delta) K_t + P_t \tag{3}$$

$$N_t = (1-\rho^x) N_{t-1} + M_t \tag{4}$$

Household takes the number of "matches", M_t, the wage rate, W_t, the rental rate R_t, and profits, P_t, as given.
FOC

$$C_t^{-\gamma} = \mathrm{E}_t \left[\beta C_{t+1}^{-\gamma} (R_{t+1} + 1 - \delta) \right]$$

Matching model example

Problem for firm matched with worker

$$\max_{k_t} z_t k_t^{\alpha} - W_t - R_t k_t^{\alpha}$$

FOC:

$$R_t = \alpha z_t k_t^{\alpha-1}$$

Firm-level profits are (at optimal k) equal to

$$p_t = (1-\alpha) z_t k_t^{\alpha} - W_t$$

Wages are given by the following rule

$$W_t = (1-\omega_0) \times [\omega_1 * p_t + (1-\omega_1)\bar{p}]$$

where $\bar{p}$ are steady state level profits. Wages are completely sticky if ω_1 is equal to 0.

Matching model example

Free entry

$$\text{posting cost} = \text{prob of success} \times \text{value if success}$$

$$\psi = \frac{M_t}{V_t} g_t$$

$$g_t = \mathrm{E}_t \left[\beta \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} p_{t+1} + (1 - \rho^x) g_{t+1} \right]$$

Matching model example

Matching technology

$$M_t = \frac{U_t V_t}{\left(U_t^{\xi} + V_t^{\xi}\right)^{1/\xi}}$$

with

$$U_t = 1 - N_{t-1}$$

Matching model example

Equilibrium

Equilibrium in the rental market

$$K_{t-1} = N_{t-1} k_t$$

profits transferred to households

$$P_t = N_{t-1} p_t - \psi V_t$$

Collecting equations: Household

$$C_t^{-\gamma} = \mathrm{E}_t \left[\beta C_{t+1}^{-\gamma} (R_{t+1} + 1 - \delta) \right]$$

exp(-nu*c)=dfactor*exp(-nu*c(+1))*(exp(r(+1))+1-delta)

$$C_t + K_t + \psi V_t = z_t K_{t-1}^{\alpha} N_{t-1}^{1-\alpha} + (1-\delta) K_t \text{ or}$$

$$C_t + I_t + \psi V_t = Y_t,\ Y_t = z_t K_{t-1}^{\alpha} N_{t-1}^{1-\alpha},\ I_t = K_t + (1-\delta) K_{t-1}$$

exp(c)+exp(i)+pcost*exp(v)=exp(y)

exp(k)=(1-delta)*exp(k(-1))+exp(i)

y=varz+alpha*k(-1)+(1-alpha)*n(-1)

Collecting equations: Matching

$$N_t = (1 - \rho^x) N_{t-1} + \frac{U_t V_t}{\left(U_t^{\xi} + V_t^{\xi} \right)^{1/\xi}}$$

```
exp(n)
= (1-rox)*exp(n(-1))+exp(u+v)
  /((exp(u*etam)+exp(v*etam))^(1/etam))
```

Collecting equations: rental rate & productivity

$$R_t = \alpha z_t k_t^{\alpha-1}$$

r=log(alpha)+varz+(alpha-1)*(k(-1)-n(-1))

$$\ln(z_t) = \rho \ln(z_{t-1}) + \varepsilon_t$$

varz=rho*varz(-1)+e

Collecting equations: free entry

$$\psi = \frac{M_t}{V_t} g_t$$

exp(eta)*exp(u)/((exp(u*etam)+exp(v*etam))^(1/etam))=pcost

$$g_t = \mathrm{E}_t \left[\beta \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} p_{t+1} + (1 - \rho^x) g_{t+1} \right]$$

exp(eta)

= dfactor*(exp(c(+1))/exp(c))^(-nu)

*(exp(prof(+1))+(1-rox)*exp(eta(+1)))

$$p_t = (1 - \alpha) z_t k_t^{\alpha} - W_t$$

prof= log((1-omega1*omega0)*(1-alpha)*exp(varz+alpha*(k(-1)-n(-1)))
-(1-omega1)*omega0*profitss)

System

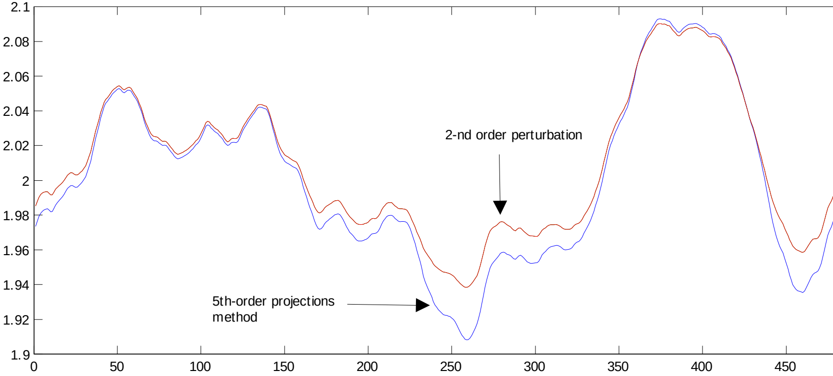
10 equations in 10 unknowns:

- $N_t, g_t, V_t, C_t, K_t, R_t, U_t, p_t, \ln(z_t), Y_t, I_t$
- n, eta, v, c, k, r, u, prof, y, varz, i

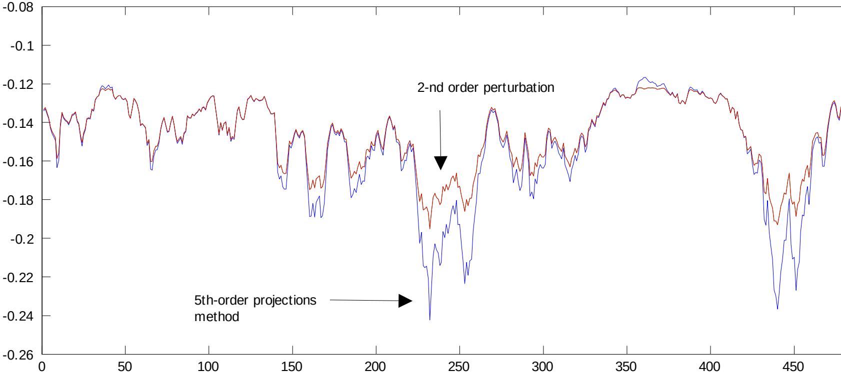
Accuracy errors

	2-nd order perturbation	5-th order projec
Capital Euler equation		
average	0.034%	0.026%
max	0.34%	0.33%
Employment Euler equation		
average	0.89%	0.004%
max	2.31%	0.086%

Log capital stock



Log employment level



Welfare-based accuracy tests

- Be careful
- Welfare loss of using $k_t = k_{ss}$ instead of the optimal policy function is relatively small