Integration for Accuracy Tests

Wouter J. Den Haan

University of Amsterdam

July 4, 2008

Numerical Integration

Quadrature techniques

- Newton-Cotes
- Gaussian-Quadrature

Quadrature techniques

Suppose we want to calculate

$$I = \int_a^b f(x)dx, \tag{1}$$

where $f(x)$ is a scalar function.
Quadrature techniques approximate I with

$$I = \int_a^b f(x)dx \approx \sum_{i=1}^{n} \omega_i f_i, \tag{2}$$

Idea behind Newton-Cotes

Approximate $f(x)$ with an n^{th}-degree polynomial and integrate this. Thus

$$\int_a^b f(x)dx \approx \int_a^b P_n(x)dx, \tag{3}$$

where $P_n(x)$ is the approximating polynomial

Beauty of Newton-Cotes

- You do not have to calculate $P_n(x)$.
- There are simple formulas that exactly implement this idea

Simpson (2nd-degree Newton-Cotes)

- $h = (b - a)/2$ (the length of the segment)
- $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$ $(= b)$
- $f_0 = f(x_0)$, $f_1 = f(x_1)$, $f_2 = f(x_2)$

$$\int_a^b f(x)dx \approx \left(\frac{1}{3}f_0 + \frac{4}{3}f_1 + \frac{1}{3}f_2\right) h, \tag{4}$$

How to make Simpson more accurate

- Split the complete interval $[a, b]$ into many small intervals with an *odd* number of nodes
- Thus, $x_0, x_1, x_2, \cdots, x_n$ and $x_i - x_{i-1} = h$
- $x_0 = a,\ x_1 = a + h,\ x_2 = a + 2h\ (= b)$

$$\begin{aligned}
& \int_a^b f(x)dx \\
= \quad & \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{n-2}^{x_n} f(x)dx \\
\approx \quad & \left(\frac{1}{3}f_0 + \frac{4}{3}f_1 + \frac{1}{3}f_2\right) h \\
& + \left(\frac{1}{3}f_2 + \frac{4}{3}f_3 + \frac{1}{3}f_4\right) h + \cdots + \left(\frac{1}{3}f_{n-2} + \frac{4}{3}f_{n-1} + \frac{1}{3}f_n\right) h
\end{aligned}$$

When to use Simpson?

- The idea behind Simpson is that the function can be approximated well locally with a second-order polynomial
- So as long as you have many nodes, it can also be used if you have nondifferentiabilities
- What if you integrate over an unbounded region?
    - You can still use Simpson as long as you can determine where the action is (trying to do this may be a good idea even if *a* and *b* are bounded

Gaussian Quadrature

Again use basic quadrature formula

$$I = \int_a^b f(x)dx \approx \sum_{i=1}^{n} \omega_i f(\zeta_i), \tag{5}$$

- Extremely powerful but meant for smooth differentiable functions.
- For example, with only 5 nodes you get an *exact* answer if $f(x)$ is a polynomial of degree 9 (or less) and an *accurate* answer if $f(x)$ is close to a 9$^{\text{th}}$-order polynomial

Gaussian Quadrature

General specification of Gaussian Quadrature

$$I = \int_a^b f(x)w(x)dx \approx \sum_{i=1}^{n} \omega_i f(\zeta_i), \tag{6}$$

- Different types of Gaussian quadrature are characterized by their choice of
    - $[a, b]$ and
    - $w(x)$ the weighting matrix.

Implementing Gaussian Quadrature

Trivial !!!!

1. Get the nodes, ζ_i, and weights for $i = 1, \cdots n$ from a subroutine
2. Calculate the function values at the nodes
3. Calculate

   $$\sum_{i=1}^{n} \omega_i f(\zeta_i), \tag{7}$$

   Note that there are no values for $w(\zeta_i)$ used. These are implicitly incorporated in the values of ω_i

Legendre Gaussian Quadrature

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} \omega_i^{GL} f(\zeta_i^{GL}).$$

When using the function legendre_quad.m you would give the following command:
[GLnodes, GLweights] = legendre_quad(GLlow, GLhigh, GLnum_nodes)

which gives GLnum_nodes nodes in the interval (GLlow,GLhigh)

Hermite Gaussian Quadrature

$$\int_{-\infty}^{\infty} f(x) e^{-x^2} dx \approx \sum_{i=1}^{n} \omega_i^{GH} f(\zeta_i^{GH}).$$

When using the function legendre_quad.m you would give the following command:

```
[GHnodes,GHweights]      = hernodes(GHnum_nodes);
```

which gives GHnum_nodes nodes

Calculating expectations (of Normally distributed variables)

- How to calculate
  $$\mathsf{E}f(y)$$
  with $x \sim N(0, \sigma^2)$.
- That is, how to calculate
  $$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} f(y) \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$
- This looks like a Gaussian Hermite problem, but is not exactly one.

Change of variables

- Let

$$y = \mu + \sqrt{2}\sigma x$$

- Then

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} f(y) \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy \\ = & \int_{-\infty}^{\infty} \left[\sqrt{2}\sigma\right] \frac{1}{\sigma\sqrt{2\pi}} f(y) \exp\left(-x^2\right) dy \\ = & \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} f(\mu + \sqrt{2}\sigma x) \exp\left(-x^2\right) dy \end{aligned}$$

where $\left[\sqrt{2}\sigma\right]$ is the Jacobian (needed because of the change in variables)

Application for accuracy test

$$
\begin{aligned}
& \mathrm{E}\left[\begin{array}{c} \beta \mathbf{c}(\; k, \exp(\rho\theta + \varepsilon')\;)^{-\gamma} \\ \times \left(\alpha \exp(\rho\theta + \varepsilon')\mathbf{k}(k_{-1}, \exp(\theta))^{\alpha-1} + 1 - \delta\right) \end{array}\right] \\
= \; & \mathrm{E}\left[\begin{array}{c} \beta \mathbf{c}(\; \mathbf{k}(k_{-1}, \exp(\theta)), \exp(\rho\theta + \varepsilon'))\;)^{-\gamma} \\ \times \left(\alpha \exp(\rho\theta + \varepsilon')\mathbf{k}(k_{-1}, \exp(\theta))^{\alpha-1} + 1 - \delta\right) \end{array}\right] \\
& \text{and} \\
\varepsilon \sim \; & N(0, \sigma^2)
\end{aligned}
$$

$$
\begin{aligned}
& \mathsf{E}\left[\begin{array}{c} \beta \mathbf{c}(\ \mathbf{k}(k_{-1}, \exp(\theta)), \exp(\rho\theta + \varepsilon'))\ )^{-\gamma} \\ \times \left(\alpha \exp(\rho\theta + \varepsilon')\mathbf{k}(k_{-1}, \exp(\theta))^{\alpha-1} + 1 - \delta\right) \end{array}\right]^{\alpha-1} \\
= \ & \int_{-\infty}^{\infty} \left[\begin{array}{c} \beta \mathbf{c}(\ \mathbf{k}(k_{-1}, \exp(\theta)), \exp(\rho\theta + \varepsilon'))\ )^{-\gamma} \\ \times \left(\alpha \exp(\rho\theta + \varepsilon')\mathbf{k}(k_{-1}, \exp(\theta))^{\alpha-1} + 1 - \delta\right) \\ \times \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(\varepsilon'-\mu)^2}{2\sigma^2}\right) \end{array} d\varepsilon' \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \left[\begin{array}{c} \beta \mathbf{c}(\ \mathbf{k}(k_{-1}, \exp(\theta)), \exp(\rho\theta + \varepsilon'))\ )^{-\gamma} \\ \times \left(\alpha \exp(\rho\theta + \varepsilon')\mathbf{k}(k_{-1}, \exp(\theta))^{\alpha-1} + 1 - \delta\right) \\ \times \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(\varepsilon' - \mu)^2}{2\sigma^2}\right) \end{array} d\varepsilon' \right] \\
&= \int_{-\infty}^{\infty} \left[\begin{array}{c} \beta \mathbf{c}(\ \mathbf{k}(k_{-1}, \exp(\theta)), \exp(\rho\theta + \sqrt{2}\sigma\tilde{\varepsilon}'))\ )^{-\gamma} \\ \times \left(\alpha \exp(\rho\theta + \sqrt{2}\sigma\tilde{\varepsilon}')\mathbf{k}(k_{-1}, \exp(\theta))^{\alpha-1} + 1 - \delta\right) \\ \times \frac{1}{\sqrt{2}} \exp\left(-\tilde{\varepsilon}^2\right) \end{array} d\varepsilon' \right]
\end{aligned}
$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\left[\begin{array}{c}\beta\mathbf{c}(\ \mathbf{k}(k_{-1},\exp(\theta)),\exp(\rho\theta+\sqrt{2}\sigma\tilde{\varepsilon}'))\ )^{-\gamma}\\ \times\left(\alpha\exp(\rho\theta+\sqrt{2}\sigma\tilde{\varepsilon}')\mathbf{k}(k_{-1},\exp(\theta))^{\alpha-1}+1-\delta\right)\ d\varepsilon'\\ \times\frac{1}{\sqrt{2}}\exp\left(-\tilde{\varepsilon}^{2}\right)\end{array}\right]\\
\approx\ \ &\sum\omega_{i}^{GH}\left[\begin{array}{c}\beta\mathbf{c}(\ \mathbf{k}(k_{-1},\exp(\theta)),\exp(\rho\theta+\sqrt{2}\sigma\zeta_{i}^{GH}))\ )^{-\gamma}\\ \times\left(\alpha\exp(\rho\theta+\sqrt{2}\sigma\zeta_{i}^{GH})\mathbf{k}(k_{-1},\exp(\theta))^{\alpha-1}+1-\delta\right)\end{array}\right]
\end{aligned}
$$

Steps

See how easy the steps are

1. Calculate $\mathbf{k}(k_{-1}, \exp(\theta))$ (or using Dynare notation $\mathbf{k}(k_{-1}, \exp(\theta_{-1}), \varepsilon))$
2. Draw the weights and nodes, ω_i^{GH} & ζ_i^{GH}
3. Calculate what the scaled nodes imply for next period's realizations of θ'
$$\exp(\rho\theta + \sqrt{2}\sigma\zeta_i^{GH}))$$
4. Calculate consumption—given the new capital stock (calculated in step 1)—at each realization of θ'
5. Calculate for each realization the term inside the sum (marginal utility of next period's consumption times the marginal product of capital)
6. weight with ω_i^{GH} and sum