Nonlinearities and Perturbation Techniques

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Nonlinearities and uncertainty in economics

- Are nonlinearities important?
- When can we expect them?

First-order perturbation and uncertainty & non-linearities

General specification

$$\mathrm{E}f(x, x', y, y') = 0.$$

- x is an $(n_x \times 1)$ vector of endogenous and exogenous state variables
- y is an $(n_y \times 1)$ vector endogenous choice variable

True solution

$$y = g(x, \sigma)$$

and

$$x' = h(x, \sigma) + \sigma\eta\varepsilon'$$

First-order perturbation and uncertainty & non-linearities

$$\max_{\{c_t, k_{t+1}\}_{t=1}^{\infty}} \mathrm{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$

$$\text{s.t.} \quad c_t + k_t = \exp(\theta_t) k_{t-1}^{\alpha} + (1-\delta) k_{t-1} \tag{1}$$

$$\theta_t = \rho \theta_{t-1} + \sigma e_t, \tag{2}$$

For notation used above: $y = c$ and $x = [k, \theta]$.

Neoclassical Growth Model

- For standard growth model we get

$$\mathsf{E}f\left([k,\theta],[k',\rho\theta+\sigma\varepsilon'],y,y'\right)=0$$

Solutions are of the form:

$$c=c(k,\theta,\sigma) \tag{3}$$

and

$$\begin{bmatrix} k' \\ \theta' \end{bmatrix} = \begin{bmatrix} k'(k,\theta,\sigma) \\ \rho\theta \end{bmatrix} + \sigma \begin{bmatrix} 0 \\ 1 \end{bmatrix} e'. \tag{4}$$

Standard first-order perturbation

Perturb around y, x, **and** σ.

$$g(x,\sigma) = g(\overline{x},0) + g_x(\overline{x},0)(x-\overline{x}) + g_\sigma(\overline{x},0)\sigma + \cdots$$

and

$$h(x,\sigma) = h(\overline{x},0) + h_x(\overline{x},0)(x-\overline{x}) + h_\sigma(\overline{x},0)\sigma + \cdots$$

Goal

Let

$$\overline{g}_x = g_x(\overline{x}, 0),\ \overline{g}_\sigma = g_\sigma(\overline{x}, 0) \text{ and}$$
$$\overline{h}_x = h_x(\overline{x}, 0),\ \overline{h}_\sigma = h_\sigma(\overline{x}, 0).$$

Goal: to find

- $(n_y \times n_x)$ matrix $\overline{g}_x$, $(n_y \times 1)$ vector $\overline{g}_\sigma$, $(n_x \times n_x)$ matrix $\overline{h}_x$, $(n_x \times 1)$ vector $\overline{h}_\sigma$.
- The total of unknowns $= (n_x + n_y) \times (n_x + 1) = n \times (n_x + 1)$.

Outcome for first-order perturbation

- Certainty equivalence: $\overline{g}_\sigma = \overline{h}_\sigma = 0$. Thus
  - changes in σ have no effect on policy rules
  - changes in σ have no effect on means of y_t and x_t.
- But
  - changes in σ do affect realizations of y_t and x_t
  - changes in σ do affect magnitude of variances and covariances
  - changes in σ does not affect correlation coefficients

How to capture non-linear effects with 1st-order perturbation: Part I

- Transformation of variables, e.g. (natural) logs

$$\begin{aligned}
\mathrm{E}f(x, x', y, y') &= \\
\mathrm{E}f(\exp(\ln(x)), \exp(\ln(x')), \exp(\ln(y)), \exp(\ln(y'))) &= \\
\mathrm{E}f(\exp(\tilde{x}), \exp(\tilde{x}'), \exp(\tilde{y}), \exp(\ln \tilde{y}')) &= \\
\mathrm{E}\tilde{f}(\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}') &= 0.
\end{aligned}$$

- What will happen with Ex and Ey if uncertainty increases?

How to capture non-linear effects with 1st-order perturbation: Part II

- Obtain a numerical approximation for as few variables as possible
- For example, for the standard growth model
  - substitute out consumption
  - obtain linear (or log-linear) law of motion for capital
  - solve for consumption using the budget constraint

$$c_t = z_t k_{t-1}^{\alpha} + (1-\delta)k_{t-1} - k(z_t, k_{t-1})$$

This is a non-linear relationship even if $k(z_t, k_{t-1})$ is linear

How to capture non-linear effects with 1st-order perturbation: Part III

Consider the following model

$$\max_{\{c_t,k_{t+1}\}_{t=1}^{\infty}} \mathrm{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma}-1}{1-\gamma}$$

$$\text{s.t.} \quad c_t + k_t = \exp(\theta_t) k_{t-1}^{\alpha} + (1-\delta) k_{t-1} \tag{5}$$

$$\theta_t = \begin{array}{c} \theta_L \text{ with probability } p(\theta) \\ \theta_H \text{ with probability } 1-p(\theta) \end{array}$$

- First-order perturbation gives the following solution

$$k_t = \overline{k} + h_k(k_{t-1} - \overline{k}) + h_\theta(\theta_t - \overline{\theta})$$

- Thus, h_k is the same independent of the value of θ

First-order conditions (allowing for autocorrelation)

- policy function when $\theta_t = \theta_L$: $k_L(k_{t-1})$
- policy function when $\theta_t = \theta_H$: $k_H(k_{t-1})$
- Euler equation when $\theta_t = \theta_L$

$$(\theta_L k_{t-1}^{\alpha} - k_{L,t})^{-\gamma} = \begin{array}{c} p_{LL}\beta(\theta_L k_{L,t}^{\alpha} - k_{L,t+1})^{-\gamma}(\alpha\theta_L k_{L,t}^{\alpha-1} + 1 - \delta) \\ (1-p_{LL})\beta(\theta_H k_{L,t}^{\alpha} - k_{H,t+1})^{-\gamma}(\alpha\theta_H k_{L,t}^{\alpha-1} + 1 - \delta) \end{array}$$

- Euler equation when $\theta_t = \theta_H$

$$(\theta_H k_{t-1}^{\alpha} - k_{H,t})^{-\gamma} = \begin{array}{c} (1-p_{HH})\beta(\theta_L k_{H,t}^{\alpha} - k_{L,t+1})^{-\gamma}(\alpha\theta_L k_{H,t}^{\alpha-1} + 1 - \delta) \\ p_{HH}\beta(\theta_H k_{H,t}^{\alpha} - k_{H,t+1})^{-\gamma}(\alpha\theta_H k_{H,t}^{\alpha-1} + 1 - \delta) \end{array}$$

- Auxiliary equation

$$k_{t-1} = (1 - z_{t-1})k_{L,t-1} + z_{t-1}k_{H,t-1}$$

- Now, θ_L & θ_H are fixed parameters and z_t is the stochastic variable.

Solving with Dynare: I

- Substitute k_{t-1} out of the two Euler equations. Now we have a standard perturbation system with three equations for $k_{L,t}$, $k_{H,t}$, and z_t.
- $k_{L,t}$ and $k_{H,t}$ have different steady state values
- Let the law of motion for z_t be given by

$$z_t - \bar{z} = \rho(z_{t-1})\left(z_{t-1} - \bar{z}\right) + \varepsilon_t. \tag{6}$$

- In this Dynare system z_t can take on any value; below we will show that this system can be made consistent with the discrete Markov process.

Solving with Dynare: II

- You could also give the linearized version of 6 to Dynare, since that is what it will be using anyway

$$z_t = \bar{z} + \rho(\bar{z})(z_{t-1} - \bar{z}) + \varepsilon_t. \tag{7}$$

with

- The unconditional mean for z_t, $\bar{z} = \mathrm{E}[z_t]$, being equal to

$$\bar{z} = \frac{1 - p_{LL}}{2 - p_{LL} - p_{HH}}\theta_H + \frac{1 - p_{HH}}{2 - p_{HH} - p_{LL}}\theta_L$$

- The unconditional mean of $\rho(z_{t-1})$, $\rho(\bar{z}) = .\mathrm{E}[\rho(z_t)]$,being equal to .

$$\bar{z} = \frac{1 - p_{LL}}{2 - p_{LL} - p_{HH}}(2p_{HH} - 1) + \frac{1 - p_{HH}}{2 - p_{HH} - p_{LL}}(2p_{LL} - 1)$$

Are Dynare specification and original model consistent?

- In simulation use 6 not 7; so you have to do your own simulation
- We need
  - $z_t \in \{0, 1\}$
  - $\mathrm{E}[\varepsilon_t | z_{t-1} = 0] = E[\varepsilon_t | z_{t-1} = 1] = 0$
  - Conditional autocorrelations have to be correct
    - $\rho(1) = 2p_{HH} - 1$
    - $\rho(0) = 2p_{LL} - 1$

Are Dynare specification and original model consistent?

- Key step is to make the distribution of ε_t dependent on the value of z_{t-1}. In particular
  - To get that

$$z_t = z_{t-1} \text{ with prob. } z_{t-1}p_{HH} + (1 - z_{t-1})p_{LL}$$

set

$$\varepsilon_t = (1 - \rho(z_{t-1}))(z_{t-1} - \bar{z}) \text{ with prob. } z_{t-1}p_{HH} + (1 - z_{t-1})p_{LL}$$

To get that

$$z_t = 1 - z_{t-1} \text{ with prob. } z_{t-1}(1 - p_{HL}) + (1 - z_{t-1})(1 - p_{LL})$$

set

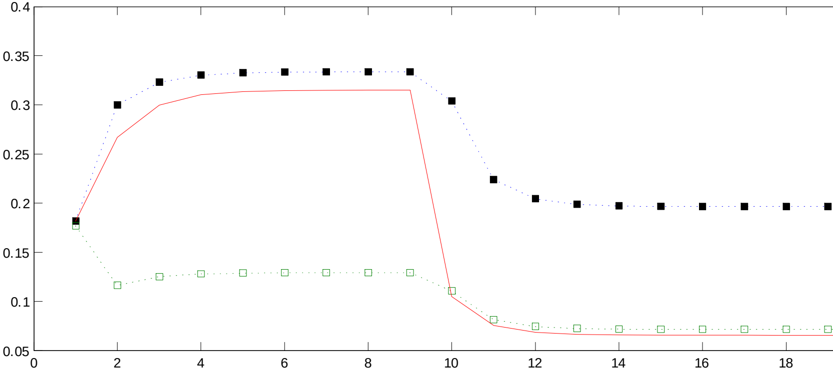
$$\varepsilon_t = -(1 + \rho(z_{t-1}))(z_{t-1} - \bar{z}) \text{ with prob. } z_{t-1}(1 - p_{HH}) + (1 - z_{t-1})(1$$

Discrete linearization - Examples - Brock-Mirman

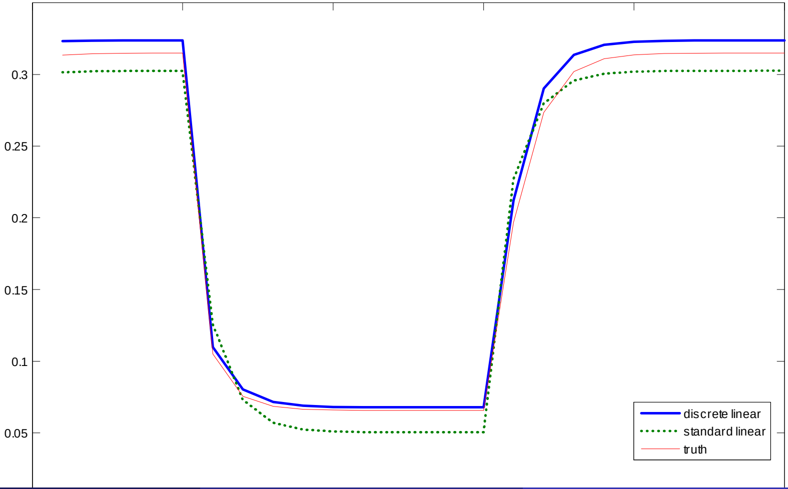
True solution is equal to

$$\ln(k_t) = \ln(\alpha\beta) + \alpha \ln(k_{t-1}) + \ln(\theta_t)$$

log-linear discrete linearization



linear discrete versus standard linearization



Second-order perturbation (scalar case & no uncertainty)

We are looking for a solution to equations like

$$f(x'', x', x) = 0 \tag{8}$$

of the form

$$x' = h(x). \tag{9}$$

Define $F(x)$ as

$$F(x) \equiv f(h(h(x)), h(x), x). \tag{10}$$

Since $h(x)$ is a solution to Equation (8), we know that

$$F(x) = 0. \tag{11}$$

Let $\overline{x}$ be the fixed-point of $h(x)$. Thus,

$$\overline{x} = h(\overline{x}). \tag{12}$$

Clearly, $\overline{x}$ has to satisfy

$$f(\overline{x}, \overline{x}, \overline{x}) = 0. \tag{13}$$

The Taylor expansion of the solution, $h(x)$, around $\overline{x}$ is given by

$$\begin{aligned} h(x) &\approx h(\overline{x}) + (x - \overline{x})h'(\overline{x}) + \frac{(x - \overline{x})^2}{2} h''(\overline{x}) + \cdots && (14) \\ &= \overline{x} + \overline{h}_1(x - \overline{x}) + \overline{h}_2 \frac{(x - \overline{x})^2}{2} + \cdots && (15) \end{aligned}$$

So the goal is to find $\overline{x}$, $\overline{h}_1$, $\overline{h}_2$, etc..

$$F(x) = 0 \ \forall x \tag{16}$$

Thus

$$F'(x) = 0 \ \forall x. \tag{17}$$

The derivative of F is given by

$$F'(x) = \frac{\partial f}{\partial x''} \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x}. \tag{18}$$

$$\left.\frac{\partial f(x'', x', x)}{\partial x''}\right|_{x''=x'=x=\overline{x}} = \overline{f}_1, \tag{19}$$

$$\left.\frac{\partial f(x'', x', x)}{\partial x'}\right|_{x''=x'=x=\overline{x}} = \overline{f}_2, \tag{20}$$

$$\left.\frac{\partial f(x'', x', x)}{\partial x}\right|_{x''=x'=x=\overline{x}} = \overline{f}_3. \tag{21}$$

Also, note that

$$\left.\frac{\partial h(x)}{\partial x}\right|_{x=\overline{x}} = \left(\overline{h}_1 + \overline{h}_2(x-\overline{x}) + \cdots\right)\Big|_{x=\overline{x}} = \overline{h}_1 \tag{22}$$

Using this in Equation (18) we get

$$F'(\overline{x}) = \overline{f}_1 \overline{h}_1^2 + \overline{f}_2 \overline{h}_1 + \overline{f}_3 = 0 \tag{23}$$

Differentiating the expresssion for $F'(x)$ in Equation (18) gives

$$F''(x) =$$
$$+\left(\frac{\partial^2 f}{\partial x''^2}\frac{\partial h(x')}{\partial x'}\frac{\partial h(x)}{\partial x}+\frac{\partial^2 f}{\partial x''\partial x'}\frac{\partial h(x)}{\partial x}+\frac{\partial^2 f}{\partial x''\partial x}\right)\left(\frac{\partial h(x')}{\partial x'}\frac{\partial h(x)}{\partial x}\right)$$
$$+\frac{\partial f}{\partial x''}\left(\frac{\partial h(x')}{\partial x'}\frac{\partial^2 h(x)}{\partial x^2}+\frac{\partial^2 h(x')}{\partial x'^2}\frac{\partial h(x)}{\partial x}\frac{\partial h(x)}{\partial x}\right)$$
$$+\left(\frac{\partial^2 f}{\partial x' x''}\frac{\partial h(x')}{\partial x'}\frac{\partial h(x)}{\partial x}+\frac{\partial^2 f}{\partial x'^2}\frac{\partial h(x)}{\partial x}+\frac{\partial^2 f}{\partial x'\partial x}\right)\frac{\partial h(x)}{\partial x}$$
$$+\frac{\partial f}{\partial x'}\frac{\partial^2 h(x)}{\partial x^2}$$
$$+\left(\frac{\partial^2 f}{\partial x x''}\frac{\partial h(x')}{\partial x'}\frac{\partial h(x)}{\partial x}+\frac{\partial^2 f}{\partial x\partial x'}\frac{\partial h(x)}{\partial x}+\frac{\partial^2 f}{\partial x^2}\right)$$

Let

$$\left.\frac{\partial^2 f(x'', x', x)}{\partial x'' \partial x}\right|_{x''=x'=x=\bar{x}} = \bar{f}_{13}. \tag{24}$$

Also,

$$\left.\frac{\partial^2 h(x)}{\partial x^2}\right|_{x=\bar{x}} = \left.\left(\bar{h}_2 + \bar{h}_3(x - \bar{x}) + \cdots\right)\right|_{x=\bar{x}} = \bar{h}_2. \tag{25}$$

Combining gives

$$\begin{aligned}F''(\overline{x}) &= 0\\ &= \left(\overline{f}_{11}\overline{h}_1^2 + \overline{f}_{12}\overline{h}_1 + \overline{f}_{13}\right)\overline{h}_1^2\\ &\quad +\overline{f}_1(\overline{h}_1\overline{h}_2 + \overline{h}_2^2\overline{h}_1)\\ &\quad + \left(\overline{f}_{21}\overline{h}_1^2 + \overline{f}_{22}\overline{h}_1 + \overline{f}_{23}\right)\overline{h}_1\\ &\quad +\overline{f}_2\overline{h}_1^2\\ &\quad + \left(\overline{f}_{31}\overline{h}_1^2 + \overline{f}_{32}\overline{h}_1 + \overline{f}_{33}\right) \qquad (26)\end{aligned}$$

Second-order perturbation with uncertainty

$$Ef(x, x', y, y') = 0. \tag{27}$$

- x is an $(n_x \times 1)$ vector of endogenous and exogenous state variables
- y is an $(n_y \times 1)$ vector endogenous choice variable.
- When applied to the stochastic growth model, we would have that $y = c$ and $x = [k, \theta]$.

$$
\begin{aligned}
h(k,\theta,\sigma) &= \bar{k} + \bar{h}_k(k-\bar{k}) + \bar{h}_\theta(\theta-\bar{\theta}) + \bar{h}_\sigma\sigma + 1/2(\\
&\quad + \bar{h}_{kk}(k-\bar{k})^2 + 2\bar{h}_{k\theta}(k-\bar{k})(\theta-\bar{\theta}) + 2\bar{h}_{k\sigma}(k-\bar{k})\sigma \\
&\quad + \bar{h}_{\theta\theta}(\theta-\bar{\theta})^2 + 2\bar{h}_{\theta\sigma}(\theta-\bar{\theta})\sigma + \bar{h}_{\sigma\sigma}\sigma^2)
\end{aligned}
$$

$$
\begin{aligned}
g(k,\theta,\sigma) &= \bar{c} + \bar{g}_k(k-\bar{k}) + \bar{g}_\theta(\theta-\bar{\theta}) + \bar{g}_\sigma\sigma + 1/2(\\
&\quad + \bar{g}_{kk}(k-\bar{k})^2 + 2\bar{g}_{k\theta}(k-\bar{k})(\theta-\bar{\theta}) + 2\bar{g}_{k\sigma}(k-\bar{k})\sigma \\
&\quad + \bar{g}_{\theta\theta}(\theta-\bar{\theta})^2 + 2\bar{g}_{\theta\sigma}(\theta-\bar{\theta})\sigma + \bar{g}_{\sigma\sigma}\sigma^2)
\end{aligned}
$$

- From the discussion above we know that $\overline{g}_{\sigma} = \overline{h}_{\sigma} = 0$
- Similarly, $\overline{g}_{k\sigma} = \overline{g}_{\theta\sigma} = \overline{h}_{k\sigma} = \overline{h}_{\theta\sigma} = 0$
- Only the constants $\overline{g}_{\sigma\sigma}$ and $\overline{h}_{\sigma\sigma}$ are affected by the amount of uncertainty.
- Does this mean that for example

$$\frac{\partial g}{\partial x}$$

is not affected by the amount of uncertainty?

Penalty function example

Point of the exercise

- Makes clear what could go wrong with second-order approximation
- Shows that first-order could be better than second-order
- Way to improve second-order approximation

Second-order perturbation

- What can change if you go from first to second-order perturbation?
  - Amount of uncertainty now does affect policy function (but only the constant)
  - Non-linear policy functions
  - Impulse response functions now depend on initial values of state variables and sign of the shocks

Model

$$\max \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma} - 1}{1-\gamma} + p(a_t)$$

s.t

$$a_t = (1+r)(y_t - c_t + a_{t-1})$$

with $p(a) < 0$ and $p'(a) > 0$

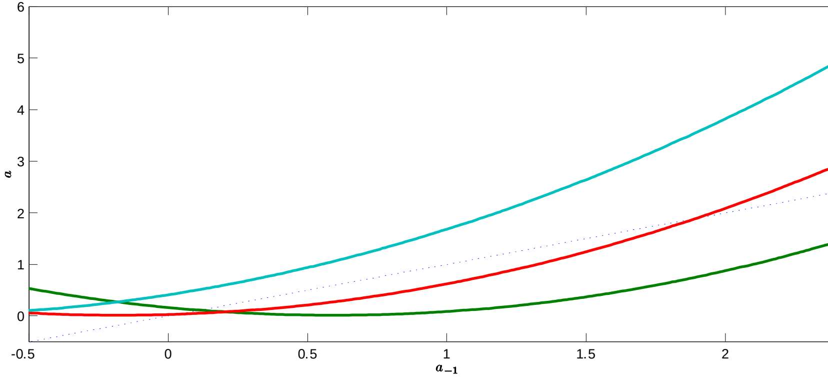
First-order condition

$$c_t^{-\gamma} = p'(a_t) + \mathrm{E}_t \left[\beta(1+r) c_{t+1}^{-\gamma} \right]$$

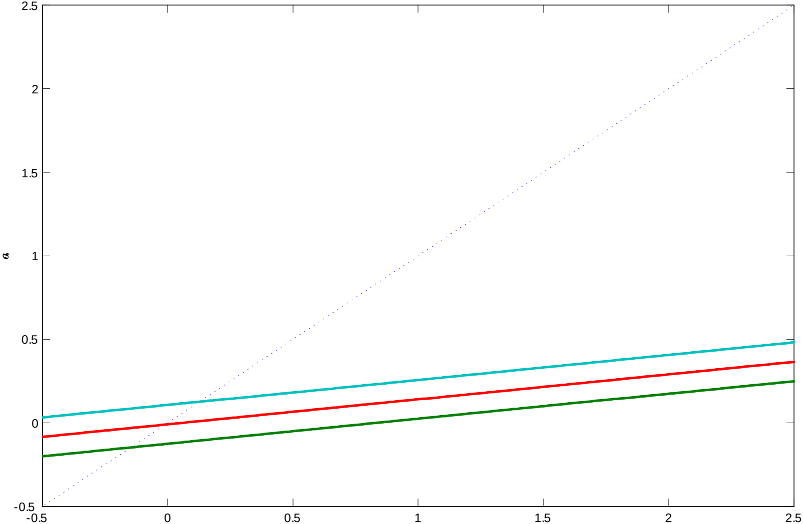
Specification

-
  - $\beta = 0.99$
  - $\gamma = 3$
  - $r = 0.03$
  - $y_t = 1.5 + \varepsilon_t$, $\varepsilon_t\tilde{}N(0, 0.15)$
  - $$ap'(a) = 0.02\exp(-70a) - 0.02$$

Second-order approximation for different values of y



First-order approximation for different values of y



Second-order approximation to penalty function