Solving Heterogeneous Agent Models with Dynare

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Individual agent

- Subject to employment, i.e., labor supply shocks:

$$e_{i,t} = \rho_e e_{i,t-1} + \varepsilon_{i,t}$$

$\varepsilon_{i,t}\tilde{}N(\mu_\varepsilon, \sigma_\varepsilon^2)$

- For now, assume μ_ε is such that aggregate labor supply equals 1, i.e.,

$$\int \exp(e_{i,t})di = 1$$

- Incomplete markets
  - only way to save is through holding capital

Individual firm

- Competitive firm so agent faces competitive prices
  - $w_t = (1-\alpha)\, z_t (K_{t-1})^{\alpha}$
  - $r_t = \alpha z_t (K_{t-1})^{\alpha-1}$

Individual agent

$$\max_{\{c_{i,t},k_{i,t}\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty} \beta^t \ln(c_{i,t})$$

$$\text{s.t.} \quad c_{i,t} + k_{i,t} = r_t k_{i,t-1} + w_t \exp(e_{i,t}) + (1-\delta)k_{i,t-1} - P(k_{i,t})$$

Penalty term

$P(k_{i,t})$ can capture many things:

- borrowing constraint: $k_{i,t} \geq 0$
- individual return is lower with lower capital holdings
- $P(k_{i,t})$ cannot be too wild if you are going to solve the model with perturbation techniques

What aggregate variables do agents care about?

- r_t and w_t
- They only depend on aggregate capital stock and z_t
- !!! This is not true in general for equilibrium prices
- Agents are interested in all information that forecasts K_t
- In principle that is the complete cross-sectional distribution of employment levels and capital levels

Equilibrium

- Continuum of agents
- Individual policy functions that solve the agent's maximization problem
- A wage and a rental rate given by equations above.
- A transition law for the cross-sectional distribution of capital, that is consistent with the investment policy function.
  - f_t = beginning-of-period cross-sectional distribution of capital and the employment status after the employment status has been realized.

$$f_{t+1} = \Upsilon(z_t, f_t)$$

Key approximating step

- Approximate the cross-sectional distribution with a limited set of characteristics
  - Proposed in Den Haan (1996), Krusell & Smith (1997,1998), Rios-Rull (1997)
- Explicitly or implicitly solve for aggregate policy rule
- Solve individual policy rule for given aggregate law of motion
- Make the two consistent

Krusell-Smith (1997,1998) algorithm

- Assume the following approximating aggregate law of motion

$$m_{t+1} = \bar{\Gamma}(z_t, m_t; \eta_{\bar{\Gamma}}),$$

  where z_t are the exogenous shocks and m_t is a vector with cross-sectional moments.

Krusell-Smith (1997,1998) algorithm

- Start with an initial guess for its coefficients, $\eta_{\bar{\Gamma}}^{0}$
- Use following iteration until $n_{\bar{\Gamma}}^{iter}$ has converged
  - Given $\eta_{\bar{\Gamma}}^{iter}$ solve for the individual policy rule.
  - Given indivudual policy rule simulate economy and generate a time series for m_t.
- Use a regression analysis to update values of η

$$\eta_{\bar{\Gamma}}^{iter+1} = \lambda \hat{\eta}_{\bar{\Gamma}} + (1-\lambda)\eta_{\bar{\Gamma}}^{iter}, \text{ with } 0 < \lambda \leq 1$$

- Iterate until $\eta_{\bar{\Gamma}}^{iter+1}$ is sufficiently close to $\eta_{\bar{\Gamma}}^{iter}$.

Solving for individual policy rules in KS algorithm

- Given aggregate law of motion you can solve for individual policy rules using your own favourite algorithm, *including Dynare*
- Note that the coefficients of the aggregate policy rules are simply parameters in the Dynare source file (the trick to redefine parameters in the Dynare source file can again be used)

Explicit aggregation (Den Haan & Rendahl)

- The simulation part is an expensive part of the KS algorithm
- In theory simulating has bad properties
- Xpa avoids the simulation step
- Xpa works for any method to solve the individual policy rule and boils down to Preston-Roca if you solve individual policy rule with perturbation

XPA first-order solution

Individual policy rule:

$$k = a_0 + a_k k_{-1} + a_e e + a_z z + a_K K_{-1},$$

Aggregation gives

$$\begin{aligned} K &= a_0 + a_k K_{-1} + a_e + a_z z + a_K K_{-1} \\ &= a_0 + a_e + (a_k + a_K) K_{-1} + a_z z \\ &= b_0 + b_K K_{-1} + b_z z \end{aligned}$$

XPA first-order solution

- Start with guess for $\{b_0, b_K, b_z\}$
- Use Dynare (first-order) to solve for $\{a_0, a_k, a_e, a_z, a_K\}$
- Update the law of motion for aggregate K:

$$b_0 = a_0 + a_e, \; b_z = a_z, \, b_K = (a_k + a_K)$$

- Iterate until convergence

XPA second-order solution

$$
\begin{aligned}
k &= a_0 + a_k k_{-1} + a_e e + a_z z + a_K K_{-1} \\
&\quad + a_{k^2} k_{-1}^2 + a_{ke} k_{-1} e + a_{kz} k_{-1} z + a_{kK} k K_{-1} \\
&\quad + a_{e^2} e^2 + a_{ez} e z + a_{eK} e K_{-1} + a_{z^2} z^2 + a_{zK} z K_{-1} + a_{K^2} K_{-1}^2
\end{aligned}
$$

Aggregation gives

$$
\begin{aligned}
K &= a_0 + a_k K_{-1} + a_e + a_z z + a_K K_{-1} \\
&\quad + a_{k^2} M(k_{-1}^2) + a_{ke} M(k_{-1} e) + a_{kz} K_{-1} z + a_{kK} K_{-1}^2 \\
&\quad + a_{e^2} M(e^2) + a_{ez} z + a_{eK} K_{-1} + a_{z^2} z^2 + a_{zK} z K_{-1} + a_{K^2} K_{-1}^2
\end{aligned}
$$

- $M(y)$ is the cross-sectional moment of variable y & $M(e^2)$ is a known constant

XPA second-order solution

Combining terms gives

$$
\begin{aligned}
K &= b_0 + b_K K_{-1} + b_z z + b_{zK} z K_{-1} + b_{K^2} K_{-1}^2 + b_{z^2} z^2 \\
&\quad + b_{M(k^2)} M(k^2) + b_{M(ke)} M(ke)
\end{aligned}
$$

$$
\begin{aligned}
b_0 &= a_0 + a_e + a_{e^2} M(e^2) \\
b_K &= a_k + a_K + a_{eK} \\
b_z &= a_z z + a_{ez} \\
b_{zK} &= +a_{kz} + a_{zK} \\
b_{K^2} &= a_{kK} + a_{K^2} \\
b_{z^2} &= a_{z^2} \\
b_{M(k^2)} &= a_{k^2} \\
b_{M(ke)} &= a_{ke}
\end{aligned}
$$

XPA second-order solution

There are now two new aggregate state variables

$$M(k_{-1}^2) \text{ and } M(k_{-1}e)$$

- If current K and r depend on these moments, then next period's values of K and r depend on next period's values of these moments. Thus, we need laws of motion to predict $M(k^2)$ and $M(ke_{+1})$
- Note that $M(ke_{+1}) = \rho_e M(ke)$

First attempt

Why not simply aggregate k^2 which is equal to

$$
\begin{aligned}
&(a_0 + a_k k_{-1} + a_e e + a_z z + a_K K_{-1} \\
&+a_{k^2} k_{-1}^2 + a_{ke} k_{-1} e + a_{kz} k_{-1} z + a_{kK} k_{-1} K_{-1} \\
&+a_{e^2} e^2 + a_{ez} ez + a_{eK} e K_{-1} + a_{z^2} z^2 + a_{zK} z K_{-1} + a_{K^2} K_{-1}^2)^2
\end{aligned}
$$

Problem:
This introduces more cross-sectional moments (and it keeps on going)

XPA trick

- Get a *second-order* approximation for k^2 and ke
- In your Dynare source file you can simply add two equations like

$$\begin{aligned} \texttt{var1} &= \texttt{k\^{}2;} \\ \texttt{var2} &= \texttt{ke;} \end{aligned}$$

- Get laws of motion for $M(k^2)$ and $M(ke_{+1})$ by aggregating the *second-order* policy functions for `var1` and `var2`.