Solving models with (lots of) idiosyncratic risk

Wouter J. Den Haan

November 14, 2009

Models with heterogeneous agents

1. Lots of idiosyncratic risk
2. If #1 leads to strong non-linearities in the policy function
   $\Longrightarrow$ high dimensional state space in hetero agents model

Outline

- Projection methods and non-linearities
- Perturbation methods and non-linearities
    - keeping problem well behaved

Projection methods - functional forms

- Splines versus polynomials
- Splines
  - be smart about grid points, e.g., log scale
- Polynomials
  - be smart about the transformation of the variables

Projection methods - finding solution

Possible choices

- Equation solver or minimization routine
  - difficult to use for splines (too many coefficients)
- Iteration procedure
  - fixed-point iteration; easier but worse convergence properties
  - time iteration; possibly a bit harder but better convergence properties

Projection methods - time versus fixed-point iteration

Both methods have in common:

- Grid for capital stock (ignore uncertainty for simplicity): $\{\kappa_j\}_{j=1}^{J}$
- Start with a guess for policy function (say consumption)
- Update policy function

Difference

- Fixed-point iteration relies more on old guess

Time versus fixed-point iteration

- fixed-point iteration

$$\frac{1}{\kappa_j^\alpha - k_{i+1}(\kappa_j)} = \frac{\beta\alpha\left[k_i(\kappa_j)\right]^{\alpha-1}}{\left(k_i(\kappa_j)\right)^\alpha - k_i(k_i(\kappa_j))}$$

  old guess used everywhere on RHS

- time iteration

$$\frac{1}{\kappa_j^\alpha - k_{i+1}(\kappa_j)} = \frac{\beta\alpha\left[k_{i+1}(\kappa_j)\right]^{\alpha-1}}{\left(k_{i+1}(\kappa_j)\right)^\alpha - k_i(k_{i+1}(\kappa_j))}$$

  old guess only used for next period's choice

Time versus fixed-point iteration

- fixed-point iteration
  - typically trivial to update
  - no guaranteed convergence; convergence often requires dampening
- time iteration
  - updating typically requires non-linear equation solver
  - guaranteed convergence (related to convergence of iterating on Bellman

Projection methods - Choosing state variable

Possible choices

- standard choice: k
- endogenous grid points: k'
  - grid for k_{+1} instead of (predetermined) k
  - solve for k instead of k_{+1}
  - note you are tracing the same function
  - makes time iteration cheap

Projection methods - Endogenous grid points

$$\frac{1}{k_{i+1}(\kappa_{+1,j}) - \kappa_{+1,j}} = \frac{\beta\alpha\kappa_{+1,j}^{\alpha-1}}{\kappa_{+1,j}^{\alpha} - k_i(\kappa_{+1,j})}$$

Perturbation and idiosyncratic risk

- Given the speed of perturbation it is naturally suited for models with heterogenous agents

Perturbation and idiosyncratic risk

- Given the speed of perturbation it is naturally suited for models with heterogenous agents
- Except ...

Perturbation and idiosyncratic risk

- Given the speed of perturbation it is naturally suited for models with heterogenous agents
- Except ...
- the non-linearities can easily create problems

Perturbation and idiosyncratic risk

- Given the speed of perturbation it is naturally suited for models with heterogenous agents
- Except ...
- the non-linearities can easily create problems
  - not just some inaccuracies but odd and explosive behavior

Perturbation and nonlinearities

- Limited radius of convergence (approximation to truth)
- Oscillating patterns; not shape preserving
- Regular polynomials: explosive behavior of dynamic systems

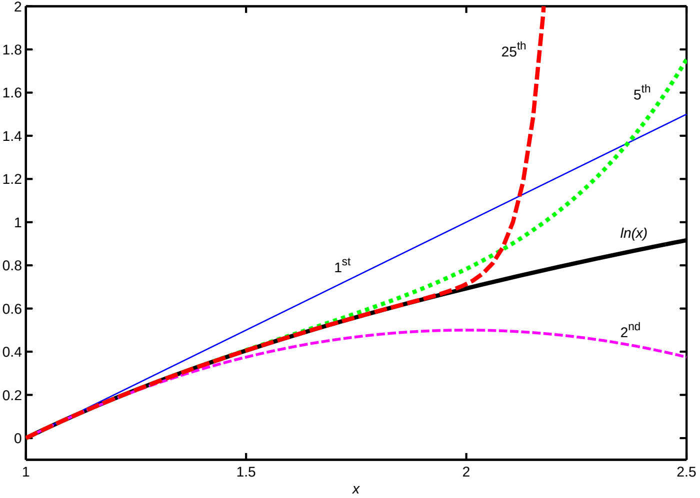
$$x_{+1} = h(x) \approx p_N(x)$$

$$\lim_{x \to \infty} \frac{\partial p_N(x)}{\partial x} = \pm\infty$$

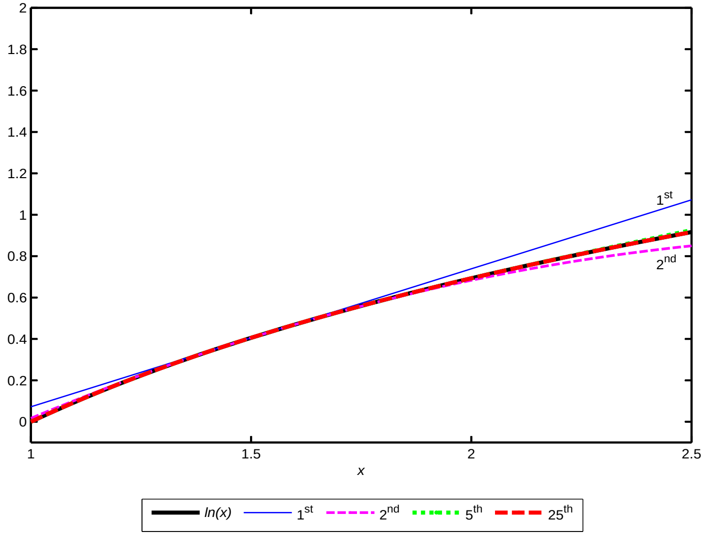
$$\text{if} \quad \lim_{x \to +\infty} \frac{\partial p_N(x)}{\partial x} = +\infty \Longrightarrow \text{no global convergence}$$

$$\text{if} \quad \lim_{x \to +\infty} \frac{\partial p_N(x)}{\partial x} = -\infty \Longrightarrow \text{function must turn negative}$$

ln(x) & Taylor series expansions at x = 1



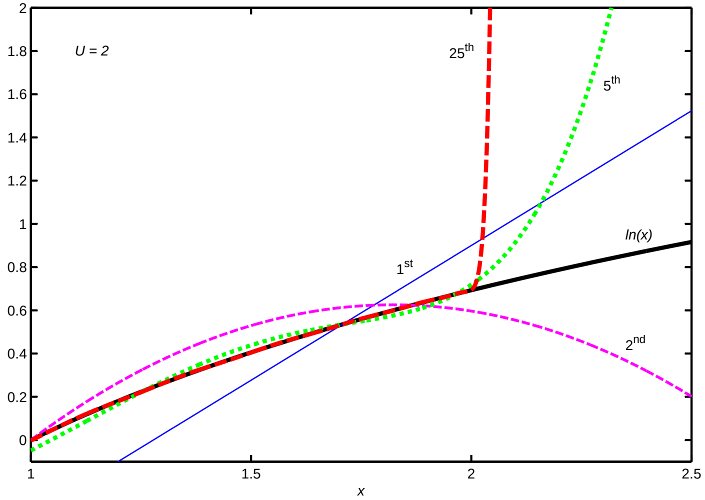
ln(x) &Taylor series expansions at x = 1.5



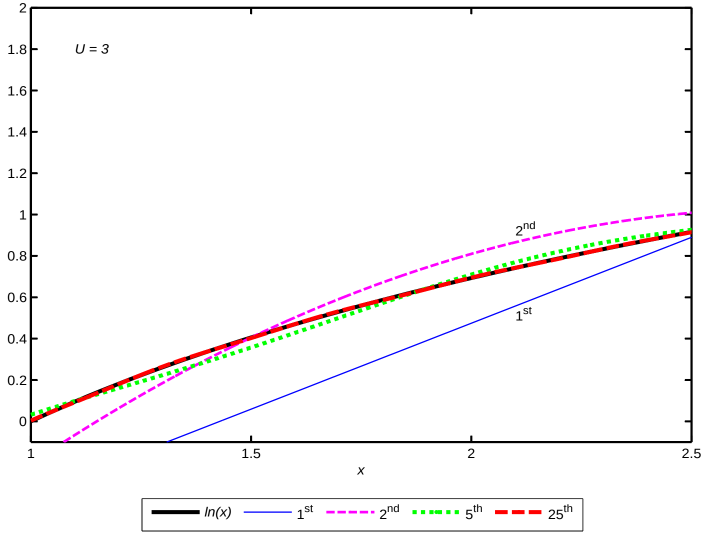
limited radius of convergence and projection methods

- Less restricted to focus on particular perturbation point
- Chebyshev nodes & compact interval $\Longrightarrow$
    - uniform convergence

ln(x) & uniform convergence in [0,2]



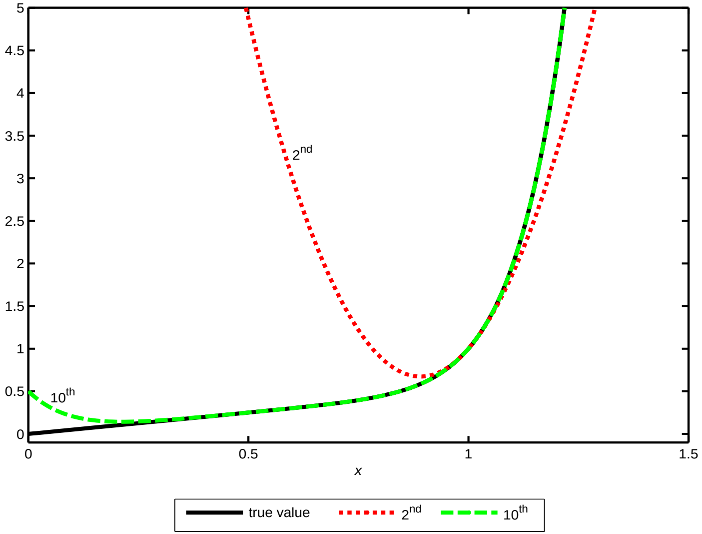
ln(x) & uniform convergence in [0,3]



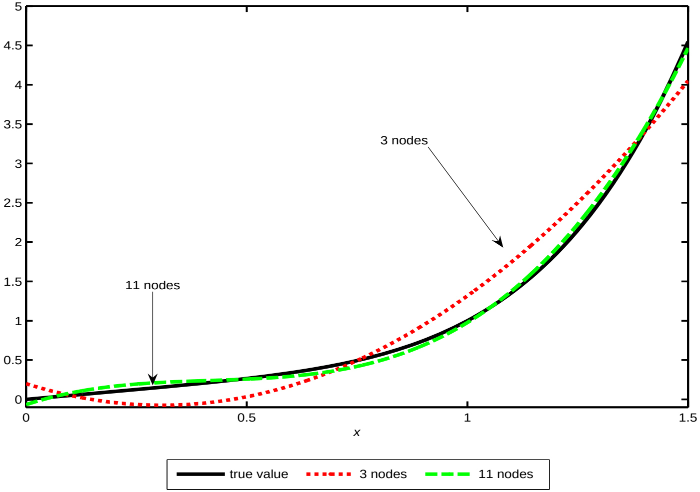
Problems within radius of convergence

- difficulties in preserving shape
- $$h(x) = 0.5x^{\alpha} + 0.5x$$
- α is an integer, so $h(x)$ is a polynomial

Perturbation solution & preserving shape



Projection solution & preserving shape

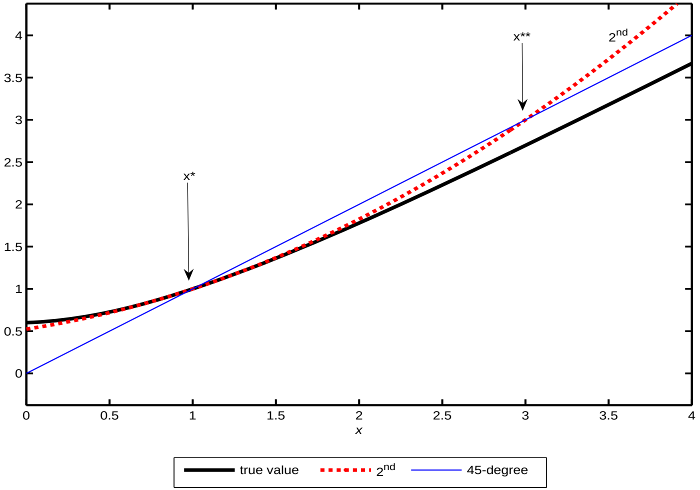


Problems within radius of convergence

- stability problems
-

$$
\begin{aligned}
a &= h(x) = \alpha_0 + x + \alpha_1 e^{-\alpha_2 x}. \\
x_{+1} &= a + \text{shock}_{+1}
\end{aligned}
$$

Perturbation solution & stability



Model

$$\max_{\{c_t,a_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\nu}-1}{1-\nu} - P(a_t)$$

s.t.

$$c_t + \frac{a_t}{1+r} = a_{t-1} + \theta_t,$$

$$\theta_t = \bar{\theta} + \varepsilon_t \text{ and } \varepsilon_t \sim N(0,\sigma^2),$$

$$a_0 \text{ given.}$$

Penalty function

Penalty function corresponding to commonly used inequality constraint:

$$P(a) = \begin{cases} \infty & \text{if } a < 0 \\ 0 & \text{if } a \geq 0 \end{cases}$$

We use:

$$P(a) = \frac{\eta_1}{\eta_0} \exp(-\eta_0 a) + \eta_2 a$$
$$\frac{\partial P(a)}{\partial a} = -\eta_1 \exp(-\eta_0 a) + \eta_2$$

if $\eta_2 > 0$ then at some point penalty function will be increasing (nice in controlling steady state levels)

Penalty function

- functional form can be approximated *globally* with Taylor series expansion
- consider different values for curvature parameter, η_0
- we do not think of penalty function as a way to implement inequality constraint
- η_1 and η_2 chosen to match mean and standard deviation of a_t

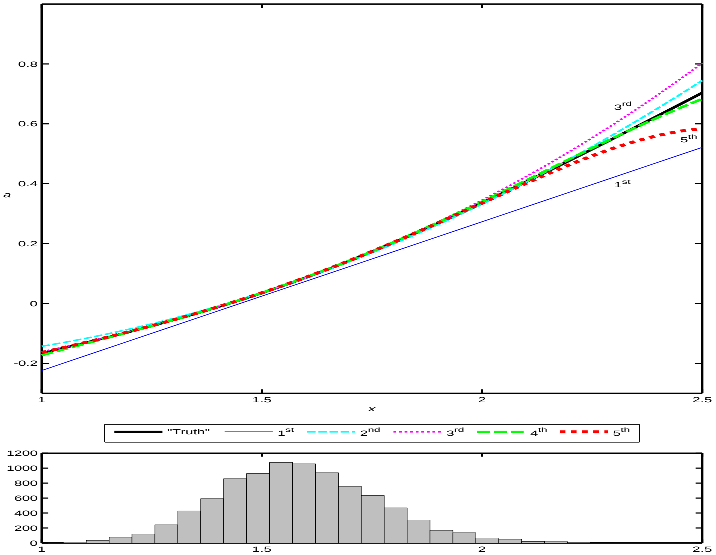
FOC

$$\frac{c_t^{-\nu}}{1+r} + \frac{\partial P(a_t)}{\partial a_t} = \beta \mathsf{E}_t \left[c_{t+1}^{-\nu} \right]$$

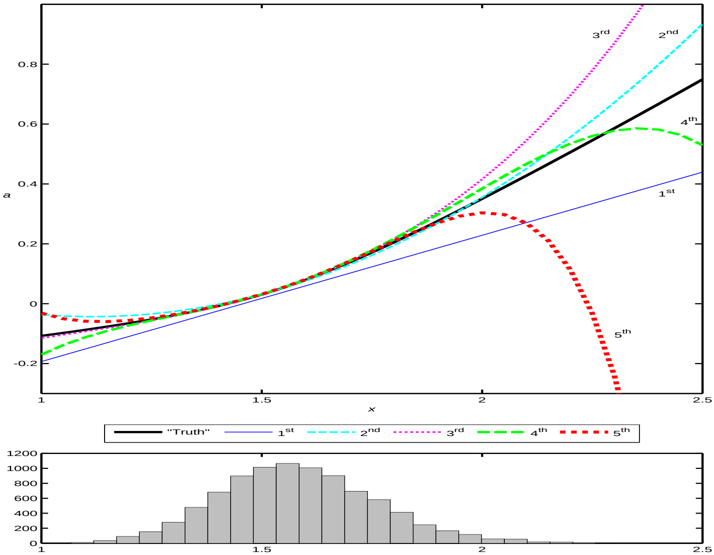
Penalty function

- we do not think of penalty function as a way to implement inequality constraint
- our calibration procedure and accurate solution $\Longrightarrow$
  - many properties of "$a \geq 0$" model similar to properties of "penalty-fcn" model

Perturbation solutions when $\eta_0 = 10$



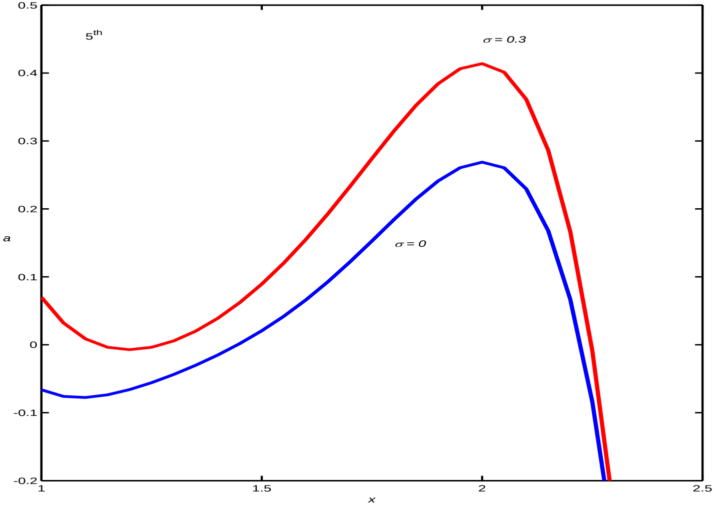
Perturbation solutions when $\eta_0 = 20$



Perturbation and higher uncertainty

- oscillations more problematic when $\sigma \uparrow$
- but higher-order perturbation solution adjust when $\sigma \uparrow$

Perturbation and more uncertainty



Simulating

- 2nd & 3rd explode
- 4th & 5th are inaccurate

Pruning - summary

- simple
- generates stable solutions for sure
- just a trick
- generates policy correspondence not function

Pruning - procedure

❶ Split up perturbation solution into two parts

$$p_{N,\text{pert}}(a_{t-1}, \theta_t) - \bar{a}_N =$$

linear part	$\gamma_{N,k}\left(a_{t-1} - \bar{a}_N\right) + \gamma_{N,\theta}\left(\theta_t - \bar{\theta}\right)$
nonlinear part	$+\tilde{p}_{N,\text{pert}}(a_{t-1} - \bar{a}_N, \theta_t - \bar{\theta})$

Pruning - procedure

2. Simulate a_t^* using

$$a_t^* - \bar{a}_N = \gamma_{N,k}\left(a_{t-1}^* - \bar{a}_N\right) + \gamma_{N,\theta}\left(\theta_t - \bar{\theta}\right).$$

Simulate $a_{\text{prune},t}$ using

$$\begin{gathered} a_{\text{prune},t} - \bar{a}_N \\ = \gamma_{N,k}\left(a_{\text{prune},t-1} - \bar{a}_N\right) + \gamma_{N,\theta}\left(\theta_t - \bar{\theta}\right) + \tilde{p}_{N,\text{pert}}(a_{t-1}^* - \bar{a}_N, \theta_t - \bar{\theta}). \end{gathered}$$

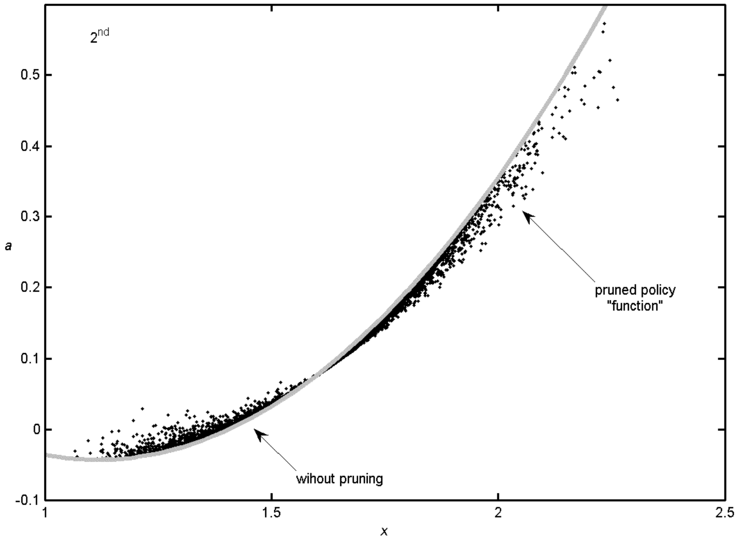
Pruning - state variables

- $a_{\text{prune},t}$ is determined by:
  - $a_{\text{prune},t-1}$ and a_{t-1}^{*}
- Thus, $a_{\text{prune},t}$ is no longer a function of the regular set of state variables

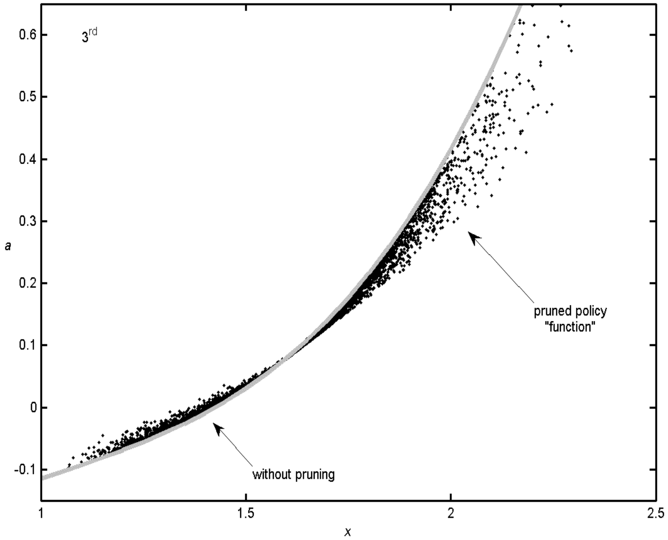
Pruning - graphs

- Our model only has one state variable, $x_t = a_{t-1} + \theta_t$
- Generate $\{a_t\}_{t=1}^{T}$ and plot simulated a_t as function of x_t

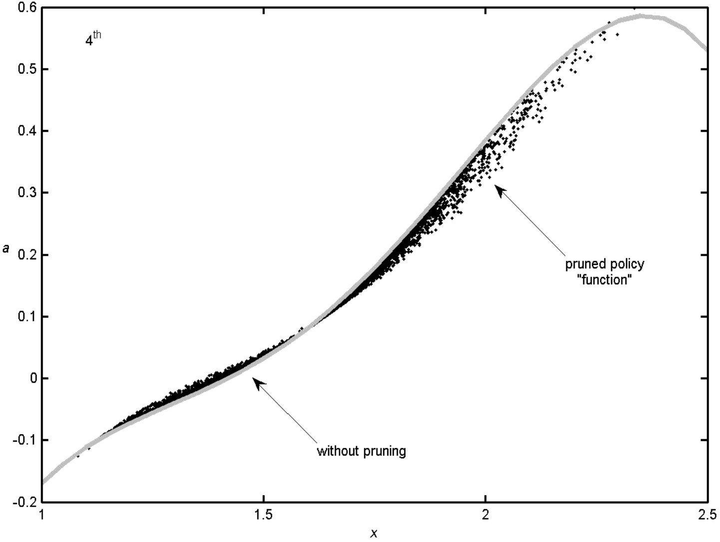
Pruning - second-order



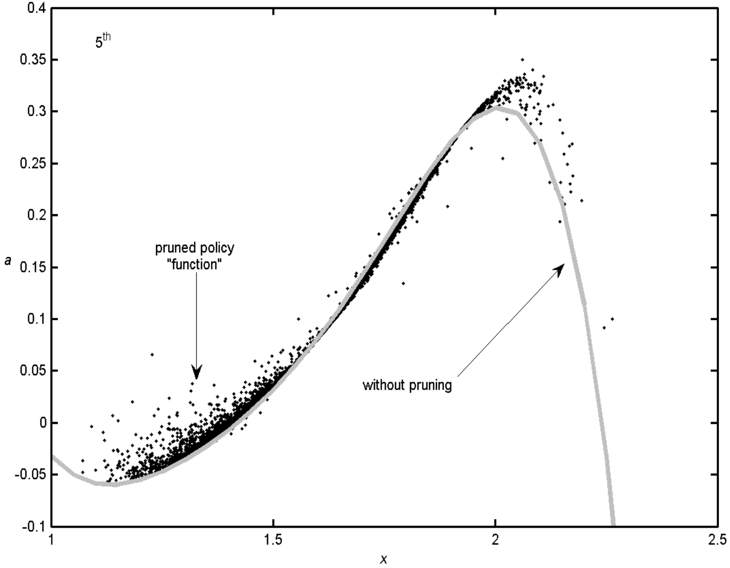
Pruning - third-order



Pruning - fourth-order



Pruning - fifth-order



General idea

Perturbation does not require you to use polynomials

- Suppose you are given

$$\left.\frac{\partial h^n(k)}{\partial k^n}\right|_{x=\bar{x}} \text{ for } n = 0, 1, \cdots, N$$

- You would like to use

$$g(k) = a_0 g_0(k) + a_1 g_1(k) + \cdots + a_N g_N(k)$$

- Solve for the values of a from the following $N+1$ equations

$$\left.\frac{\partial h^n(k)}{\partial k^n}\right|_{k=\bar{k}} = [a_0, a_1, \cdots, a_N] \begin{bmatrix} \left.\frac{\partial g_0^n(k)}{\partial k^n}\right|_{k=\bar{k}} \\ \vdots \\ \left.\frac{\partial g_N^n(k)}{\partial k^n}\right|_{k=\bar{k}} \end{bmatrix}$$

Trivial example

$$1/x$$

- Fourth-order Taylor series expansion

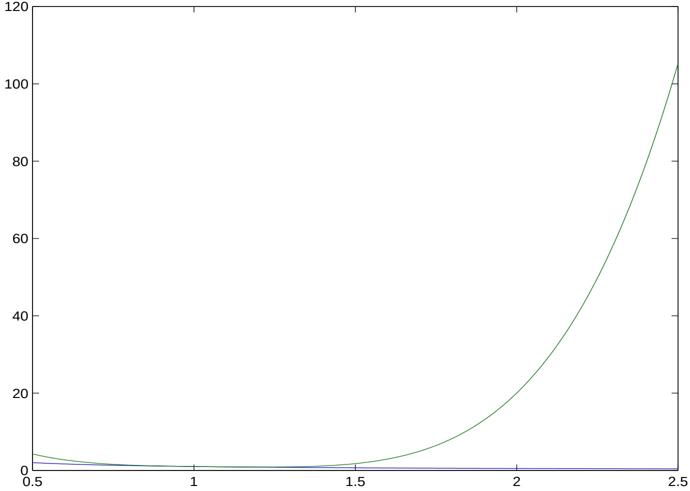
$$1/x \approx 1 - (x-1) + 2(x-1)^2 - 6(x-1)^3 + 24(x-1)^4$$

- Alternative

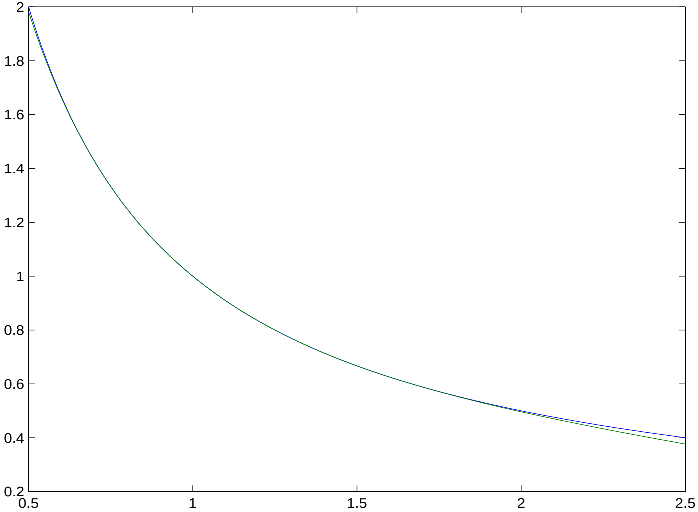
$$1/x \approx a_0 e^{-2x} + a_1 e^{-2x} x + a_2 e^{-2x} x^2 + a^3 e^{-2x} x^3 + a^4 e^{-2x} x^4$$

- note that this is not a transformation

Standard Taylor expansion



Alternative Taylor expansion



Properties of DSGE models

- The true solution of DSGE models typically satisfy
  - monotoniciy
  - stability
- Can perturbation be modified to impose this?

Solutions

All solutions proposed satisfy the following

1. Use smooth differentiable functions
2. Satisfy the perturbation principle

Solutions considered

1. Change of variables
2. Other basis functions
3. True shape preserving (work in progress)

Change of variables - idea

$p_N(x)$: regular N^{th}-order perturbation solution

$$
\begin{aligned}
x_{+1} &= p_N(x) \\
\tilde{x}_{+1} &= x_{+1} - p_1(x) \\
\tilde{x}_{+1} &= \frac{2\bar{\gamma}^*}{1+\exp(-\hat{x}_{+1})} - \bar{\gamma}^*
\end{aligned}
$$

Change of variables - idea

- From
$$p_N(x) - p_1(x) = \frac{2\bar{\gamma}^*}{1+\exp(-\hat{x}_{+1})} - \bar{\gamma}^*$$
obtain regular perturbation solution for $\hat{x}$
- Use as the alternative perturbation solution

$$x_{+1} = p_1(x) + \tilde{x}_{+1} \approx p_1(x) + \frac{2\bar{\gamma}^*}{1+\exp(-\hat{p}_N(x))} - \bar{\gamma}^*.$$

Change of variables -idea

Properties

- For $\bar{\gamma}^*$ small enough close to $p_1(x)$ and, thus, monotone and stable

Change of variables - implementation

- Standard perturbation system (with $p_1(k)$ given)

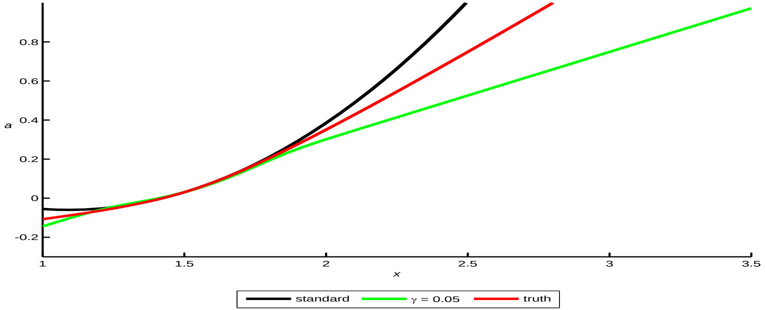
$$\frac{1}{\left(k^\alpha + (1-\delta)k - k_{+1}\right)^\nu} = \frac{\beta\left(\alpha k_{+1}^{\alpha-1} + 1 - \delta\right)}{\left(k_{+1}^\alpha + (1-\delta)k_{+1} - k_{+2}\right)^\nu},$$

$$k_{+1} - p_1(k) = \frac{2\bar{\gamma}^*}{1+\exp(-\hat{k}_{+1})} - \bar{\gamma}^*,$$

- Use

$$k_{+1} = p_1(k) + \frac{2\bar{\gamma}^*}{1+\exp(-\hat{p}_N(k))} - \bar{\gamma}^*.$$

2nd order



True underlying problem solved?

- Problem with standard polynomial basis functions
  - higher-order terms always dominate away from steady state
- Still true for

$$x_{+1} \approx p_1(x) + \frac{2\bar{\gamma}^*}{1 + \exp(-\hat{p}_N(x))} - \bar{\gamma}^*$$

- Extension: use separate squashing function for each basis order

Alternative basis functions

First-order:

$$b_1(x;\gamma_{1,N}) = \gamma_{1,N}(x-\bar{x}).$$

Higher-order:

$$b_n(x;\gamma_{n,N},\gamma^*_{n,N}) = \frac{2\gamma^*_{n,N}}{1+\exp\left\{-\frac{\gamma_{n,N}}{n!\gamma^*_{n,N}}\left(x-\bar{x}\right)^n\right\}} - \gamma^*_{n,N}$$

Simple version:

$$b_n(x;\gamma_n,\gamma^*) = \frac{2\gamma^*}{1+\exp\left\{-\frac{\gamma_n}{n!\gamma^*}\left(x-\bar{x}\right)^n\right\}} - \gamma^*$$

Properties basis functions

1. **Zero property** at $x = \bar{x}$.

$$\forall N \text{ and } \forall n \leq N \text{ we have } b_n(0; \gamma_n, \gamma^*) = 0.$$

2. Levels are bounded from above and below.
3. Derivatives at $x = \bar{x}$ do not depend on γ^*
   - $\Longrightarrow$ solutions for γ_n coefficients do not depend on γ^*
4. Even-order basis functions are not monotonic
5. γ_n could have the wrong sign!

Properties approximation

1. x_t cannot go to infinity for any choice of γ^*
2. γ^* can always be chosen small enough to ensure monotonicity and a unique fixed point
3. Letting γ^* depend on n and N obviously has advantages
   - reduce $\gamma^*_{n,N}$ when n is odd or if $b_{n,N}$ has the wrong sign

Solving for the coefficients

More properties of the basis functions

$$\left.\frac{\partial^i b_n(x;\gamma_{n,N},\gamma^*_{n,N})}{x^i}\right|_{x=\bar{x}} = 0 \text{ for } i < n$$
$$\left.\frac{\partial^i b_n(x;\gamma_{n,N},\gamma^*_{n,N})}{x^i}\right|_{x=\bar{x}} \neq 0 \text{ for } i > n$$

This is enough to solve coefficient recursively

Solving for the coefficients

$$\begin{aligned}
\bar{h}_1 &= \gamma_{1,N} \\
\bar{h}_2 &= \gamma_{2,N} \left.\frac{\partial^2 b_2(x;\cdot,\cdot)}{\partial x^2}\right|_{x=\bar{x}} \\
\bar{h}_3 &= \gamma_{2,N} \left.\frac{\partial^3 b_2(x;\cdot,\cdot)}{\partial x^3}\right|_{x=\bar{x}} + \gamma_{3,N} \left.\frac{\partial^3 b_3(x;\cdot,\cdot)}{\partial x^3}\right|_{x=\bar{x}} \\
&\vdots \\
\bar{h}_N &= \gamma_{2,N} \left.\frac{\partial^N b_2(x;\cdot,\cdot)}{\partial x^N}\right|_{x=\bar{x}} + \gamma_{3,N} \left.\frac{\partial^N b_3(x;\cdot,\cdot)}{\partial x^N}\right|_{x=\bar{x}} \\
&\quad + \cdots + \gamma_{N,N} \left.\frac{\partial^N b_N(x;\cdot,\cdot)}{\partial x^N}\right|_{x=\bar{x}}
\end{aligned}$$

Multivariate version

x is a $J \times 1$ vector

First-order

$$b_{j,1}(x; \Gamma_{j,1,N}) = \Gamma'_{j,1,N}(x - \bar{x})$$

Higher-order

$$b_{j,n}(x; \Gamma_{n,N}, \gamma^*_{n,N}) = \frac{2\gamma^*_{j,n,N}}{1 + \exp\left\{-\frac{1}{n!\gamma^*_{j,n,N}} Q_n\left(x; \Gamma_{j,n,N}\right)\right\}} - \gamma^*_{j,n,N}$$

- $Q_n(x, \Gamma_{j,n,N})$ is a n^{th}-order polynomial basis function of the vector x with coefficients $\Gamma_{j,n,N}$.

Multivariate version - 2nd order

$$b_{j,1}(x;\Gamma_{j,1,2}) = \Gamma'_{j,1,2}(x-\bar{x}),$$

$$b_{j,2}(x;\Gamma_{j,2,2},\gamma^*_{j,2,2})$$
$$= \frac{2\gamma^*_{j,2,2}}{1+\exp\left\{-\frac{1}{2!\gamma^*_{j,2,2}}(x-\bar{x})'\,\Gamma_{j,2,2}(x-\bar{x})\right\}} - \gamma^*_{j,2,2},$$

Perturbation point and lots of idiosyncratic risk

- With idioysncratic risk variable can move far away from perturbation point
  - e.g., in KS model perturbation point for employed and unemployed the same
  - but it doesn't have to be

Consider the following model

$$\max_{\{c_t,k_{t+1}\}_{t=1}^{\infty}} \mathsf{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma}-1}{1-\gamma}$$

$$\text{s.t.} \;\; c_t + k_t = \exp(\theta_t) k_{t-1}^{\alpha} + (1-\delta) k_{t-1}$$

$$\theta_{t+1} = \begin{array}{c} \theta_L \text{ with probability } p(\theta|\theta_t) \\ \theta_H \text{ with probability } 1 - p(\theta_t) \end{array}$$

- First-order perturbation:

$$k_t = \bar{k} + h_k(k_{t-1} - \bar{k}) + h_\theta(\theta_t - \bar{\theta})$$

- Thus, h_k is the same independent of the value of θ

First-order conditions

- policy function when $\theta_t = \theta_L$: $k_L(k_{t-1})$
- policy function when $\theta_t = \theta_H$: $k_H(k_{t-1})$
- Now, θ_L & θ_H are fixed parameters and z_t is the stochastic variable.

Euler equations when $\theta_t = \theta_L$ and $\theta_t = \theta_H$ and auxiliary equation

$$\begin{array}{c} (\theta_L k_{t-1}^{\alpha} - k_{L,t})^{-\gamma} \\ = \begin{array}{c} p_{LL}\beta(\theta_L k_{L,t}^{\alpha} - k_{L,t+1})^{-\gamma}(\alpha\theta_L k_{L,t}^{\alpha-1} + 1 - \delta) \\ (1 - p_{LL})\beta(\theta_H k_{L,t}^{\alpha} - k_{H,t+1})^{-\gamma}(\alpha\theta_H k_{L,t}^{\alpha-1} + 1 - \delta) \end{array} \end{array}$$

$$\begin{array}{c} (\theta_H k_{t-1}^{\alpha} - k_{H,t})^{-\gamma} \\ = \begin{array}{c} (1 - p_{HH})\beta(\theta_L k_{H,t}^{\alpha} - k_{L,t+1})^{-\gamma}(\alpha\theta_L k_{H,t}^{\alpha-1} + 1 - \delta) \\ p_{HH}\beta(\theta_H k_{H,t}^{\alpha} - k_{H,t+1})^{-\gamma}(\alpha\theta_H k_{H,t}^{\alpha-1} + 1 - \delta) \end{array} \end{array}$$

$$k_{t-1} = (1 - z_{t-1})k_{L,t-1} + z_{t-1}k_{H,t-1}$$

New system with new variables

- Substitute out k_{t-1}. Now z_t enters the orginal Euler equations
- $k_{L,t}$ and $k_{H,t}$ have different steady state values
- Let the law of motion for z_t be given by

$$z_t - \bar{z} = \rho(z_{t-1})\,(z_{t-1} - \bar{z}) + \varepsilon_t. \tag{1}$$

$$\begin{aligned} & \left(\theta_L \left((1-z_{t-1})k_{L,t-1} + z_{t-1}k_{H,t-1}\right)^{\alpha} - k_{L,t}\right)^{-\gamma} \\ = \quad & p_{LL}\beta(\theta_L k_{L,t}^{\alpha} - k_{L,t+1})^{-\gamma}(\alpha\theta_L k_{L,t}^{\alpha-1} + 1 - \delta) \\ & (1-p_{LL})\beta(\theta_H k_{L,t}^{\alpha} - k_{H,t+1})^{-\gamma}(\alpha\theta_H k_{L,t}^{\alpha-1} + 1 - \delta) \end{aligned}$$

$$\begin{aligned} & \left(\theta_H \left((1-z_{t-1})k_{L,t-1} + z_{t-1}k_{H,t-1}\right)^{\alpha} - k_{H,t}\right)^{-\gamma} \\ = \quad & (1-p_{HH})\beta(\theta_L k_{H,t}^{\alpha} - k_{L,t+1})^{-\gamma}(\alpha\theta_L k_{H,t}^{\alpha-1} + 1 - \delta) \\ & p_{HH}\beta(\theta_H k_{H,t}^{\alpha} - k_{H,t+1})^{-\gamma}(\alpha\theta_H k_{H,t}^{\alpha-1} + 1 - \delta) \end{aligned}$$

$$z_t - \bar{z} = \rho(z_{t-1})\left(z_{t-1} - \bar{z}\right) + \varepsilon_t$$

- You could also use the linearized version of 1, since that is what will be using anyway

$$z_t = \bar{z} + \rho(\bar{z})(z_{t-1} - \bar{z}) + \varepsilon_t. \tag{2}$$

with

- The unconditional mean for z_t, $\bar{z} = \mathsf{E}[z_t]$, being equal to

$$\bar{z} = \frac{1 - p_{LL}}{2 - p_{LL} - p_{HH}}\theta_H + \frac{1 - p_{HH}}{2 - p_{HH} - p_{LL}}\theta_L$$

- The unconditional mean of $\rho(z_{t-1})$, $\rho(\bar{z}) = .\mathsf{E}[\rho(z_t)]$,being equal to .

$$\bar{z} = \frac{1 - p_{LL}}{2 - p_{LL} - p_{HH}}(2p_{HH} - 1) + \frac{1 - p_{HH}}{2 - p_{HH} - p_{LL}}(2p_{LL} - 1)$$

Are new and original model consistent?

- In simulation use 1 not 2; so you have to do your own simulation
- We need
  - $z_t \in \{0,1\}$
  - $\mathsf{E}[\varepsilon_t|z_{t-1}=0] = E\left[\varepsilon_t|z_{t-1}=1\right] = 0$
  - Conditional autocorrelations have to be correct
    - $\rho(1) = 2p_{HH} - 1$
    - $\rho(0) = 2p_{LL} - 1$

Are new and original model consistent?

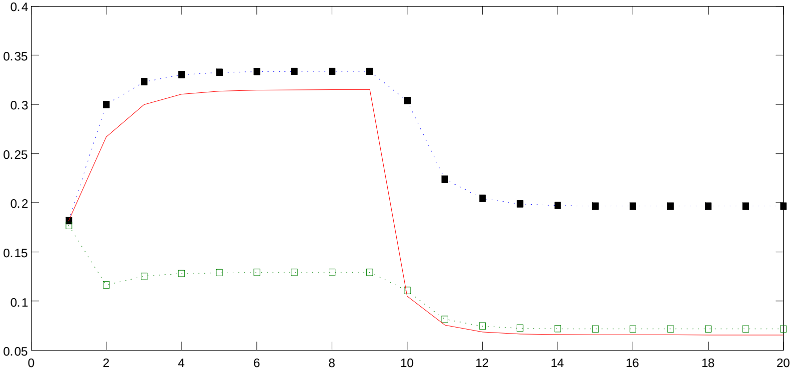
to get that $z_t = z_{t-1}$ with prob $z_{t-1}p_{HH} + (1 - z_{t-1})p_{LL}$

set $\varepsilon_t = (1 - \rho(z_{t-1}))(z_{t-1} - \bar{z})$
with prob $z_{t-1}p_{HH} + (1 - z_{t-1})p_{LL}$

to get that $z_t = 1 - z_{t-1}$
with prob $z_{t-1}(1 - p_{HL}) + (1 - z_{t-1})(1 - p_{LL})$

set $\varepsilon_t = -(1 + \rho(z_{t-1}))(z_{t-1} - \bar{z})$
with prob $z_{t-1}(1 - p_{HH}) + (1 - z_{t-1})(1 - p_{LL})$

log-linear discrete linearization



linear discrete versus standard linearization