

Solving Models with Heterogeneous Agents Applications

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Models considered

- Monetary models with consumer heterogeneity
- Models with entrepreneurs
 - discrete choice
- Turning KS into a matching model
- Portfolio problem

Monetary model I: simple MIU model

- Endowment economy
- Idiosyncratic and aggregate endowment shocks
- Agents can hold money to ensure themselves
- Money also provides liquidity services
- Equilibrium model but constant money supply (for now)

Exogenous processes

Endowment process:

$$y_{i,t} = a_t \theta \exp(e_{i,t})$$

$$a_t = (1 - \rho_a) + \rho_a a_{t-1} + \varepsilon_{a,t}, \quad \varepsilon_{a,t} \sim N(0, \sigma_a^2)$$

$$e_{i,t} = \rho_e e_{i,t-1} + \varepsilon_{e,i,t} \sim N(0, \sigma_e^2)$$

$$\int y_{i,t} di = a_t$$

Household problem

$$\begin{aligned} \max_{\{c_{i,t}, M_{i,t}\}_{t=1}^{\infty}} \quad & \sum_{t=1}^{\infty} \beta^{t-1} \left[\ln c_{i,t} + \mu \ln \frac{M_{i,t}}{p_t} \right] \\ \text{s.t.} \quad & P_t c_{i,t} + M_{i,t} = P_t y_{i,t} + M_{i,t-1} \\ & M_{i,0} \text{ given} \end{aligned}$$

First-order condition

$$\frac{1}{P_t} \frac{1}{c_{i,t}} = \mathbb{E}_t \left[\beta \frac{1}{P_{t+1}} \frac{1}{c_{i,t+1}} + \frac{\mu}{M_{i,t}} \right]$$

Equilibrium

$$\int M_{i,t} di = \int \bar{M} di = \bar{M}$$

(note that there is a unit mass of agents)

Getting started

- Give the true set of state variables
 - they have to be known at the beginning of the period
- Which cross-sectional moments would be included if you include second-order moments
- Why would $\rho_e = 0$ reduce the computational burden?

Quick (and dirty) solution

Procedure

- 1 Solve the rep. agent model
- 2 Use its law of motion to obtain solution to our model

Rep. agent

$$\frac{1}{P_t} \frac{1}{a_t} = E_t \left[\beta \frac{1}{P_{t+1}} \frac{1}{a_{t+1}} + \frac{\mu}{M} \right]$$

Alternative strategy

- Start with $P_t = \gamma_0 + \gamma_1 a$
- Using this one can solve for $M_{i,t}$ from

$$\frac{1}{P_t} \frac{1}{c_{i,t}} = E_t \left[\beta \frac{1}{P_{t+1}} \frac{1}{c_{i,t+1}} + \frac{\mu}{M_{i,t}} \right]$$
$$P_t c_{i,t} + M_{i,t} = P_t y_{i,t} + M_{i,t-1}$$
$$P_t = \gamma_0 + \gamma_1 a$$

Problem

- In simulation you will never get

$$\int M_{i,t} = \bar{M}$$

even if approximation for P_t is quite good

- Even worse, errors are likely to accumulate \implies
Disequilibrium is accumulating over time
- Not the type of errors you want to live with

Solution to the problem

- Solve for $M_i(P_t, e_{i,t}, a_t)$ instead of $M_i(P_t, e_{i,t}, a_t)$
- Each period solve for P_t from

$$\int M_i(P_t, e_{i,t}, a_t) = \bar{M}$$

Solution to the problem

- Which $M_i(P_t, e_{i,t}, a_t)$ to choose?
- Lots probably work
 - !!! But you do need

$$\frac{\partial M_i(P_t, \cdot)}{\partial P_t} > 0$$

Procedure

- Start with $P_t = \gamma_0 + \gamma_1 a$
- In addition to the model

$$\begin{aligned}\frac{1}{P_t} \frac{1}{c_{i,t}} &= \mathbb{E}_t \left[\beta \frac{1}{P_{t+1}} \frac{1}{c_{i,t+1}} + \frac{\mu}{M_{i,t}} \right] \\ P_t c_{i,t} + M_{i,t} &= P_t y_{i,t} + M_{i,t-1} \\ P_t &= \gamma_0 + \gamma_1 a\end{aligned}$$

define $P_{i,t}$ using

$$M_{i,t} = P_t + \bar{M} - P_{i,t}$$

Procedure

- This gives following numerical solutions:
 - $c_i(e_{i,t}, a_t), M_i(e_{i,t}, a_t), P_i(e_{i,t}, a_t)$
- Do **not** use:
 - $c_i(e_{i,t}, a_t), M_i(e_{i,t}, a_t)$
- Only use

$$M_{i,t} = P_t + \bar{M} - P(e_{i,t}, a_t)$$

Procedure

- Solve for P_t from

$$\int M_{i,t} di = \int (P_t + \bar{M} - P(e_{i,t}, a_t)) di$$

- Since

$$\int M_{i,t} di = \int \bar{M} di$$

we get

$$P_t = \int P_{i,t} di$$

Procedure

- Update γ_0 and γ_1
 - With X_{pa} coefficients would follow directly from

$$P_t = \int P_{i,t} di$$

- With KS you would run a regression

Model with entrepreneurs - discrete choice

- ex ante identical agents
- perpetual youth model (constant probability of dying)
- agents can buy ability to invest in more productive capital (become entrepreneurs)
- all agents work (for simplicity)

Representative firm

$$Y_t = a_t (K_t + K_{e,t})^\alpha L_t^{1-\alpha}$$

$$r_t = \alpha a_t (K_t + K_{e,t})^{\alpha-1} L_t^{1-\alpha}$$

$$w_t = (1 - \alpha) a_t (K_t + K_{e,t})^\alpha L_t^{-\alpha}$$

Entrepreneur

$$v_e(e_{i,t}k_{i,t}, a_t) = \max_{k_{i,t+1}} \left(\ln(r_t e_{i,t} k_{i,t} + (1 - \delta)e_{i,t} k_{i,t} + w_t - k_{i,t+1}) + \beta(1 - \rho) \mathbf{E}_t [v_e(k_{i,t+1}, a_{t+1})] \right)$$

$$e_{i,t} \sim N(1 + \mu, \sigma_e^2), \quad \mu > 0$$

Entrepreneur - FOC

$$\frac{1}{c_{i,t}} = \mathbb{E}_t \left[\frac{\beta e_{i,t+1} (r_{t+1} + (1 - \delta))}{c_{i,t+1}} \right]$$

Non-entrepreneur

$$\begin{aligned}
 & v(\varepsilon_{i,t}k_{i,t}, a_t) \\
 & = \\
 \max & \left\{ \begin{array}{l} \max_{k_{i,t+1}} \left(\ln(r_t \varepsilon_{i,t} k_{i,t} + (1 - \delta) \varepsilon_{i,t} k_{i,t} + w_t - k_{i,t+1}) \right. \\ \qquad \qquad \qquad \left. + \beta(1 - \rho) \mathbb{E}_t [v(k_{i,t+1}, a_{t+1})] \right) \\ \max_{k_{i,t+1}} \left(\ln(r_t \varepsilon_{i,t} k_{i,t} + (1 - \delta) \varepsilon_{i,t} k_{i,t} + w_t - k_{i,t+1} - \psi) \right. \\ \qquad \qquad \qquad \left. + \beta(1 - \rho) \mathbb{E}_t [v_e(k_{i,t+1}, a_{t+1})] \right) \end{array} \right\}
 \end{aligned}$$

$$\varepsilon_{i,t} \sim N(1, \sigma_e^2)$$

Non-entrepreneur - FOC

$$\frac{1}{c_{i,t}} = \mathbb{E}_t \left[\frac{\beta \varepsilon_{i,t+1} (r_{t+1} + (1 - \delta))}{c_{i,t+1}} \right]$$

level $\bar{\varepsilon}k_t$ pinned down by

$$\max_{k_{i,t+1}} \left(\begin{array}{c} \ln(r_t \bar{\varepsilon}k_t + (1 - \delta)\bar{\varepsilon}k_t + w_t - k_{i,t+1}) \\ + \beta(1 - \rho) \mathbb{E}_t [v(k_{i,t+1}, a_{t+1})] \end{array} \right)$$

=

$$\max_{k_{i,t+1}} \left(\begin{array}{c} \ln(r_t \bar{\varepsilon}k_t + (1 - \delta)\bar{\varepsilon}k_t + w_t - k_{i,t+1} - \psi) \\ + \beta(1 - \rho) \mathbb{E}_t [v_e(k_{i,t+1}, a_{t+1})] \end{array} \right)$$

Equilibrium

$$N_{e,t+1} = (1 - \rho)N_{e,t} + F_t(\bar{\varepsilon}k_t)$$

$$K_t = N_{e,t} \int k dF_{e,t} + (1 - N_{e,t}) \int k dF_{e,t}$$

$$L_t = 1$$

Getting started

- Give the true set of state variables
- How to deal $F_t(\bar{\epsilon k}_t)$?
- With which aggregate laws of motion could you solve the individual problem?

The KS model with matching

- Unemployment rate is exogenous in KS
- More interesting would be to make this endogenous using a matching model

Individual agent

- Subject to employment shocks ($e_{i,t} \in \{0, 1\}$)
- constant wage rate and no unemployment insurance
- Incomplete markets
 - only way to save is through holding ownership shares
 - penalty function on number of shares

Laws of motion

- a_t can take on two values
- e_t can take on two values as in original KS model
 - but probabilities determined endogenously

Individual agent

$$\max_{\{c_{i,t}, s_{i,t+1}\}_{t=0}^{\infty}} E \sum_{t=0}^{\infty} \beta^t (\ln(c_{i,t}) - F(s_{i,t+1}))$$

s.t.

$$c_{i,t} + s_{i,t+1}p_t = we_{i,t} + s_{i,t}(p_t + d_t)$$

- simple problem for given process of r_t and matching probabilities

Penalty function

$$F(s_{i,t+1}) = \frac{\eta_1}{\eta_0} \exp(-\eta_0 s_{i,t+1}) + \eta_1 \exp(1) s_{i,t+1}$$

$$f(s_{i,t+1}) = \frac{\partial F(s_{i,t+1})}{\partial s_{i,t+1}} = -\eta_1 \exp(-\eta_0 s_{i,t+1}) + \eta_1 \exp(1)$$

Minimum of penalty function

$$f(s_{i,t+1}) = \frac{\partial F(s_{i,t+1})}{\partial s_{i,t+1}} = 0 \text{ at } s_{i,t+1} = 1$$

- changes in $s_{i,t+1}$ costly if η_0 high
- If η_0 high enough, then
 - $s_{i,t+1} \approx 1$
 - $c_{i,t} \approx we_{i,t} + d_t$

Individual agent - FOC condition

$$\frac{p_t}{c_{i,t}} + f(s_{i,t+1}) = \mathbf{E}_t \left[\frac{\beta (p_{t+1} + d_{t+1})}{c_{i,t+1}} \right]$$

Law of motion of employment status

$$\text{prob}(e_{+1} = 1 | e = 1) = 1 - \rho$$

$$\text{prob}(e_{+1} = 1 | e = 0) = \pi_t$$

π_t is endogenous

Law of motion of employment status

$$\text{prob}(e_{+1} = 1 | e = 1) = 1 - \rho$$

$$\text{prob}(e_{+1} = 1 | e = 0) = \pi_t$$

π_t is endogenous

Representative firm

$$\max_{\{n_{t+1}, k_t, v_t\}_{t=1}^{\infty}} \mathbf{E}_1 \left[\sum_{t=1}^{\infty} \beta^{t-1} \frac{c_1}{c_t} ((a_t - w) n_t - \psi v_t) \right]$$

s.t.

$$n_{t+1} = \tilde{\pi}_t v_t + (1 - \rho_x) n_t$$

- $\tilde{\pi}_t$: matching probability vacancy
- MRS used is ad hoc. Why?

Representative firm - FOC

$$\psi = \tilde{\pi}_t \lambda_t$$

$$\lambda_t = \mathbb{E}_t [\beta (a_{t+1} - w + (1 - \rho_x) \lambda_{t+1})]$$

Equilibrium and matching market

$$\int s_{i,t} di = 1 \text{ or } c_t = \int c_{i,t} di = w_t n_t + d_t$$

$$m_t = v_1 (1 - n_t)^{v_2} v_t^{1-v_2}$$

$$\tilde{\pi}_t = \frac{m_t}{v_t} \quad \& \quad \pi_t = \frac{m_t}{1 - n_t}$$

Getting started

- Give the true set of state variables
- which laws of motions for aggregate variables do you need
- how connected are the different sectors?

Static portfolio problem

$$\max_{\alpha} E [U(\alpha r_1 + (1 - \alpha)r_2)]$$

$$r_1 \sim N(\mu, \sigma_1^2)$$

$$r_2 \sim N(\mu, \sigma_2^2)$$

Steady state value of α (i.e. value when $\sigma_1 = \sigma_2 = 0$) is not defined

Devereux-Sutherland trick

Solve α from second-order approximation

$$\begin{aligned} & [U(\alpha r_1 + (1 - \alpha)r_2)] \\ & \quad \approx \\ & \quad U(\mu) \\ & \quad + \alpha U'(\mu)(r_1 - r_2) \\ & \quad + 0.5\alpha^2 U''(\mu)((r_1 - \mu)^2 + (r_2 - \mu)^2) \end{aligned}$$

Devereux-Sutherland trick

Solve α from second-order approximation

$$\max_{\alpha} E0.5(\alpha^2(r_1 - \mu)^2 + (1 - \alpha)^2(r_2 - \mu)^2)$$

$$\alpha = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

But is this a difficult problem?

- Using standard projection techniques this is a trivial problem

$$\max_{\alpha} \frac{\sum_{i_1}^I \sum_{i_2}^I U \left(\begin{array}{c} \alpha \left(\sigma_1 \sqrt{2} \zeta_{i_1} \right) \\ + (1 - \alpha) \left(\sigma_2 \sqrt{2} \zeta_{i_2} \right) \end{array} \right) \omega_1 \omega_2}{\sqrt{\pi}}$$

Making portfolio problem well behaved

- Portfolio problems remain a bit tricky
- Start with a problem with frictions so that problem is trivial to solve
- Gradually decrease friction
- Ask whether friction should be zero to answer your question

Making portfolio problem well behaved

$$\max_{\alpha} -\eta_0 (\alpha - \bar{\alpha})^2 + \mathbb{E}[U(\alpha r_1 + (1 - \alpha)r_2)]$$

- Problem easier when $\eta_0 > 0$
- Many insights same for $\eta_0 = 0$ and $\eta_0 > 0$
- For example

$$\frac{\partial \alpha}{\partial \sigma_1} < 0$$

Dynamic portfolio problem

- log endowment

$$y_t = \rho_y y_{t-1} + e_{y,t} \quad e_{y,t} \sim N(0, \sigma_y^2)$$

- returns

$$r_f \sim N(\mu_{r_f}, 0)$$

$$r \sim N(\mu_r, \sigma_r^2)$$

- Budget constraint

$$c_t + s_t = y_t + (1 + \alpha_{t-1} r_f + (1 - \alpha_{t-1}) r_t) s_{t-1}$$

Maximization problem

$$\max_{\{\alpha_t, s_t, c_t\}} \sum_{t=1}^{\infty} \beta^{t-1} \left(\begin{array}{l} \ln(c_t) - \frac{\theta_0}{2} (\alpha_t - 0.5)^2 \\ -\frac{\eta_1}{\eta_0} * \exp(-\eta_0 s_t) - \eta_1 s_t \end{array} \right)$$

- first-order conditions

$$\frac{1}{c_t} = \mathbb{E} \left[\frac{\beta (1 + \alpha_t r_f + (1 - \alpha_t) r_{t+1})}{c_{t+1}} \right] + \eta_1 (\exp(-\eta_0 s_t) - s_t)$$

$$0 = \mathbb{E} \left[\frac{r_f - r_{t+1}}{c_{t+1}} \right] - \theta_0 (\alpha_t - 0.5)$$

- How to start?