

Solving Models with Heterogeneous Agents - KS algorithm

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Individual agent

- Subject to employment shocks ($e_{i,t} \in \{0, 1\}$)
- Incomplete markets
 - only way to save is through holding capital
 - borrowing constraint $k_{i,t+1} \geq 0$

Laws of motion

- z_t can take on two values
- e_t can take on two values
- probability of being (un)employed depends on z_t
- transition probabilities are such that unemployment rate only depends on current z_t
- Thus $u_t = u^b$ if $z_t = z^b$ and $u_t = u^g$ if $z_t = z^g$ with $u^b > u^g$.

Individual agent

$$\max_{\{c_{i,t}, k_{i,t+1}\}_{t=0}^{\infty}} E \sum_{t=0}^{\infty} \beta^t \ln(c_{i,t})$$

s.t.

$$c_{i,t} + k_{i,t+1} = r_t k_{i,t} + (1 - \tau_t) w_t \bar{l} e_{i,t} + \mu w_t (1 - e_{i,t}) + (1 - \delta) k_{i,t}$$
$$k_{i,t+1} \geq 0$$

for given process of r_t and w_t , this is a relatively simple problem

What aggregate variables do agents care about?

- r_t and w_t
- They only depend on aggregate capital stock and z_t
- !!! This is not true in general for equilibrium prices
- Agents are interested in all information that forecasts K_t
- In principle that is the complete cross-sectional distribution of employment status and capital levels

Equilibrium - first part

- Individual policy functions solving agent's max problem
- A wage and a rental rate given by equations above.

Equilibrium - second part

- A transition law for the cross-sectional distribution of capital, that is consistent with the investment policy function.

$$f_{t+1} = Y(z_{t+1}, z_t, f_t)$$

- f_t = beginning-of-period cross-sectional distribution of capital and the employment status after the employment status has been realized.
- z_{t+1} does not affect the cross-sectional distribution of capital but does affect the joint cross-sectional distribution of capital and employment status

Key approximating step

- Approximate the cross-sectional distribution with a limited set of characteristics
 - Proposed in Den Haan (1996), Krusell & Smith (1997,1998), Rios-Rull (1997)
- Solve for aggregate policy rule
- Solve individual policy rule for a given aggregate law of motion
- Make the two consistent

Krusell-Smith (1997,1998) algorithm

- Assume the following approximating aggregate law of motion

$$m_{t+1} = \bar{\Gamma}(z_{t+1}, z_t, m_t; \eta_{\bar{\Gamma}}).$$

- Start with an initial guess for its coefficients, $\eta_{\bar{\Gamma}}^0$

Krusell-Smith (1997,1998) algorithm

- Use following iteration until $n_{\bar{\Gamma}}^{iter}$ has converged
 - Given $\eta_{\bar{\Gamma}}^{iter}$ solve for the individual policy rule.
 - Given individual policy rule simulate economy and generate a time series for m_t .
- Use a regression analysis to update values of η

$$\eta_{\bar{\Gamma}}^{iter+1} = \lambda \hat{\eta}_{\bar{\Gamma}} + (1 - \lambda) \eta_{\bar{\Gamma}}^{iter}, \quad \text{with } 0 < \lambda \leq 1$$

Individual policy rules & projection

- Given aggregate law of motion you can solve for individual policy rules using your own favourite algorithm
- As easy as solving representative agent model, except that number of state variables has increased
- State variables for agent: $s_{i,t} = \{k_{i,t}, e_{i,t}, a_t\}$ and $a_t = \{z_t, m_t\} = \{z_t, K_t, \tilde{m}_t\}$.

Individual policy rules & projection

- a_t must "reveal" K_t
 - $a_t \implies K_t \implies r_t^k$ and r_t^w
- Let $K_{t+1} = \bar{\Gamma}_K(z_{t+1}, z_t, a_t; \eta_{\bar{\Gamma}})$, $\tilde{m}_{t+1} = \bar{\Gamma}_{\tilde{m}}(z_{t+1}, z_t, a_t; \eta_{\bar{\Gamma}})$
- If a_t includes many characteristics of the cross-sectional distribution \implies high dimensional individual policy rule

Individual policy rules & projection methods

- As always, choose which function to approximate
- here

$$k_{i,t+1} = P_n(s_{i,t}; \eta_{P_n})$$

Individual policy rules & projection methods

- $\{s_K\}_{K=1}^{\mathcal{K}}$ the set with \mathcal{K} nodes
- $s_K = \{k_K, e_K, a_K\}$, and $a_K = \{z_K, K_K, \tilde{m}_K\}$

Individual policy rules & projection methods

- First-order condition

$$\left(\begin{array}{c} (r^k(z_\kappa, K_\kappa) + 1 - \delta)k_\kappa \\ +(1 - \tau(z_\kappa))r^w(z_\kappa, K_\kappa)\bar{l}e_\kappa + \mu r^w(z_\kappa, K_\kappa)(1 - e_\kappa) \\ -P_n(s_\kappa; \eta_{P_n}) \end{array} \right)^{-\nu}$$

$$= \mathbb{E} \left[\begin{array}{c} \beta(r^k(z', K') + (1 - \delta)) \times \\ (r^k(z', K') + 1 - \delta)P_n(s_\kappa; \eta_{P_n}) \\ +(1 - \tau(z'))r^w(z', K')\bar{l}e' + \mu r^w(z', K')(1 - e') \\ -P_n(s'; \eta_{P_n}) \end{array} \right)^{-\nu}$$

Individual policy rules & projection methods

Euler equation errors:

$$u_{\kappa} = \left(\begin{array}{c} (r^k(z_{\kappa}, K_{\kappa}) + 1 - \delta)k_{\kappa} \\ +(1 - \tau(z_{\kappa}))r^w(z_{\kappa}, K_{\kappa})\bar{l}e_{\kappa} + \mu r^w(z_{\kappa}, K_{\kappa})(1 - e_{\kappa}) \\ -P_n(s_{\kappa}; \eta_{P_n}) \end{array} \right)^{-v} -$$

$$\sum_{z' \in \{z^b, z^g\}} \sum_{e' \in \{0, 1\}} \left[\begin{array}{c} \beta(r^k(z', K') + (1 - \delta)) \times \\ \left(\begin{array}{c} (r^k(z', K') + 1 - \delta)P_n(s_{\kappa}; \eta_{P_n}) \\ +(1 - \tau(z'))r^w(z', K')\bar{l}e' \\ +\mu r^w(z', K')(1 - e') \\ -P_n(s'; \eta_{P_n}) \end{array} \right)^{-v} \\ \times \\ \pi(e', z' | z_{\kappa}, e_{\kappa}) \end{array} \right]$$

Error depends on known variables and η_{P_n} when using

$$r^k(z_\kappa, K_\kappa) = \alpha z_\kappa (K_\kappa / L(z_\kappa))^{\alpha-1}$$

$$r^w(z_\kappa, K_\kappa) = (1 - \alpha) z_\kappa (K_\kappa / L(z_\kappa))^\alpha$$

$$r^k(z', K') = \alpha z' (K' / L(z'))^{\alpha-1}$$

$$= \alpha z' (\bar{\Gamma}_K(z', z_\kappa, a_\kappa; \eta_{\bar{\Gamma}}) / L(z'))^{\alpha-1}$$

$$r^w(z', K') = (1 - \alpha) z' (K' / L(z'))^\alpha$$

$$= (1 - \alpha) z' (\bar{\Gamma}_K(z', z_\kappa, a_\kappa; \eta_{\bar{\Gamma}}) / L(z'))^\alpha$$

$$\tau(z) = \mu(1 - L(z)) / \bar{l}L(z)$$

$$s' = \left\{ \begin{array}{c} k', e', z', \\ \bar{\Gamma}_K(z', z_\kappa, a_\kappa; \eta_{\bar{\Gamma}}), \bar{\Gamma}_{\bar{m}}(z', z_\kappa, a_\kappa; \eta_{\bar{\Gamma}}) \end{array} \right\}$$

Again standard projection problem

- Find η_{P_n} by minimizing the errors

Remaining issues

- Using just the mean and approximate aggregation
- Simulation
- Other models and always ensuring equilibrium

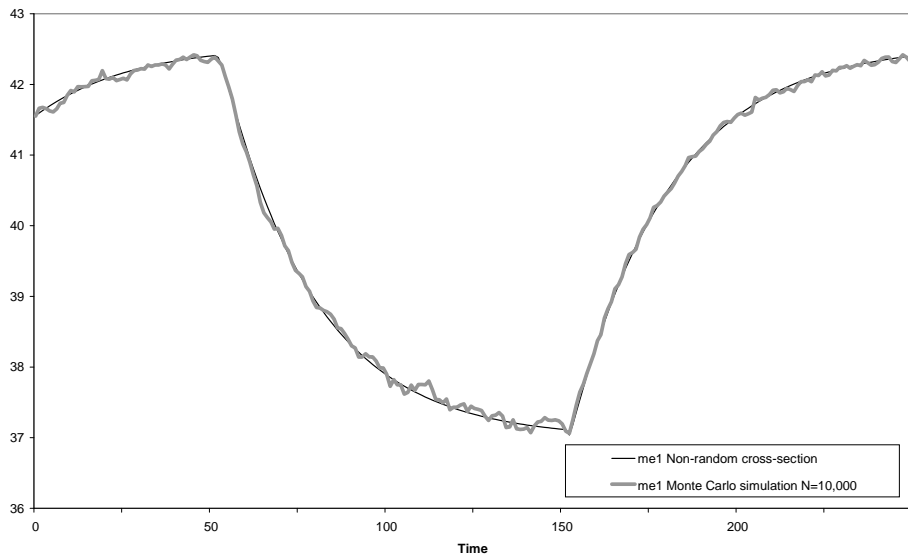
Approximate aggregation

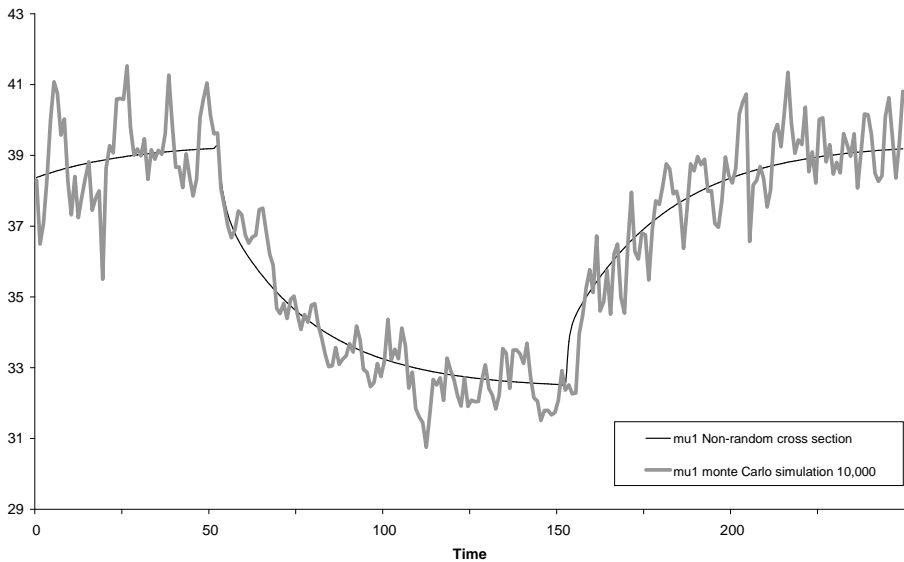
- The mean is often sufficient \Rightarrow close to complete markets
- Why does only the mean matter?

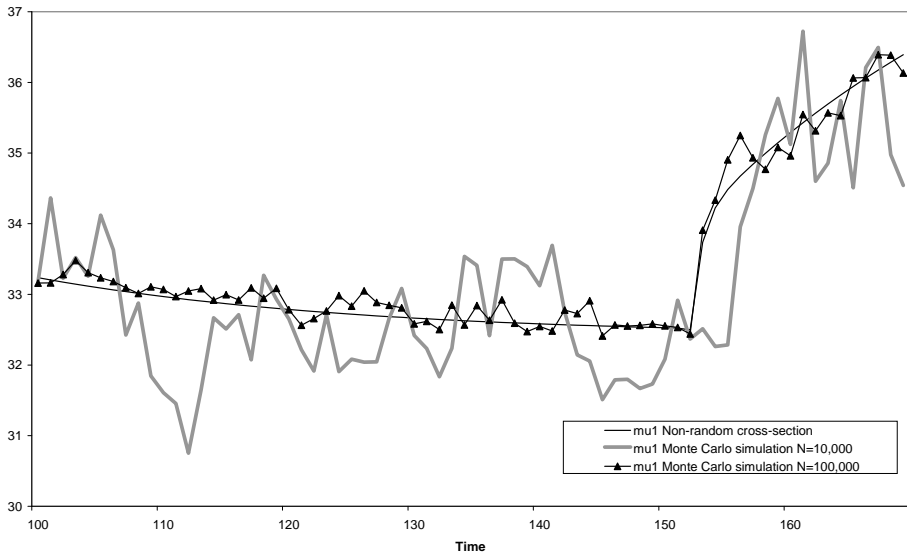
How to simulate?

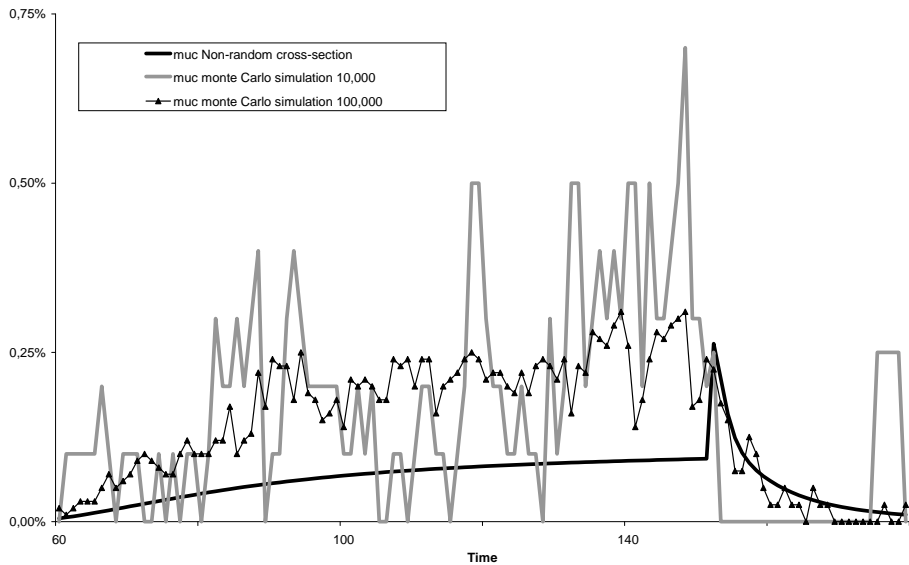
- Numerical procedure with a continuum of agents.
- What if you really do like to simulate a panel with a finite number of agents?
 - Impose truth as much as possible: if you have 10,000 agents have exactly 400 (1,000) agents unemployed in a boom (recession)
 - Even then sampling noise is non-trivial

Simulation and sampling noise









Imposing equilibrium

- In model above, equilibrium is automatically imposed in simulation
- Why?

Imposing equilibrium

- What if we add one-period bonds?
 - Also solve for
 - individual demand for bonds, $b(s_{i,t})$
 - bond price $q(a_t)$
 - In simulation aggregate demand for bonds is not necessarily equal to zero
 - Why is this problematic?

Bonds and ensuring equilibrium I

- Add the bond price as a state variable in individual problem
 - a bit weird (making endogenous variable a state variable)
 - risky

Bonds and ensuring equilibrium II

- Don't solve for

$$b_i(s_{i,t})$$

- but solve for

$$b_i(q_t, s_{i,t})$$

- Solve q_t from

$$0 = \left(\sum b_i(q_t, s_{i,t}) \right) / I$$

Bonds and ensuring equilibrium II

- How to get $b_i(q_t, s_{i,t})$?

- 1 Solve for $d_i(s_{i,t})$ where

$$d(s_{i,t}) = b(s_{i,t}) + q(a_t)$$

- 2 Imposing equilibrium gives

$$\begin{aligned} 0 &= \left(\sum b_i(q_t, s_{i,t}) \right) / I \quad \implies \\ q_t &= \left(\sum d_i(s_{i,t}) \right) / I \\ b_{i,t+1} &= d(s_{i,t}) - q(a_t) \end{aligned}$$

Bonds and ensuring equilibrium II

- Does any $b_i(q_t, s_{i,t})$ work?
- For sure it needs to be a demand equation, that is

$$\frac{\partial b_i(q_t, s_{i,t})}{\partial q_t} < 0$$

KS algorithm: Advantages & Disadvantages

- simple
- MC integration to calculate cross-sectional means
 - can easily be avoided
- Points used in projection step are clustered around the mean
 - Theory suggests this would be bad (recall that even equidistant nodes does not ensure uniform convergence; Chebyshev nodes do)
 - At least for the model in KS (1998) this is a non-issue; in comparison project the aggregate law of motion for K obtained this way is the most accurate