Solving Models with Heterogeneous Agents (not with KS or Xpa)

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Other algorithms

- Den Haan (1996)
  - very simple
- Roca & Preston (2007)
  - pure perturbation, thus fast
- Algan, Allais, & Den Haan (2008)
  - pure projection; can handle transition from non-typical cross-sectional distributions
- Reiter (2008)
  - smart hybrid; projection to deal with idiosyncratic risk, perturbation to keep cost low

Den Haan 1996

- Cross-sectional distribution characterized with finite set of moments
- No *explicit* approximate law of motion for aggregate variables
  - so also no additional inaccuracies introduced
- Full simulation method so has all the disadvantages of simulation methods

Simulating the panel

Conditional expectation of individual approximated with $p_N(s_t;\psi)$

$$c_{i,t}^{1-\nu} = p_N(s_{i,t};\psi)$$

solve $k_{i,t+1}$ from budget constraint

$$\begin{array}{ll} \text{if } k_{i,t+1} \geq 0 & \text{done} \\ \text{if } k_{i,t+1} < 0 & \left\{ \begin{array}{c} k_{i,t+1} = 0 \\ \text{solve } c_{i,t} \text{ from bc} \end{array} \right. \end{array}$$

Simulating the panel for given value psi

Steps:

- start in $t = 1$ with cross-section of I agents.
- use cross-section to calculate K_t and other moments that are part of $s_{i,t}$
- use K_t and a_t to calculate r_t and w_t
- for each agent calculate $k_{i,t+1}$
- go to the next period

Simulating the panel

- To calculate aggregate variables you need a panel
- To update individual problem, you only need to store aggregate variables and individual variables of 1 agent

Updating the value of psi

True model:

$$\text{if } k_{t+1} > 0 \quad \begin{array}{c} c_t^{-\nu} = \mathrm{E}_t\left[\beta c_{t+1}^{-\nu}(r_{t+1} + 1 - \delta)\right] \\ \text{solve } k_{t+1} \text{ from budget constraint} \end{array}$$

- regress $\beta c_{t+1}^{-\nu}(r_{t+1} + 1 - \delta)$ on $p_n(s_t; \psi)$ for those observations when $k_{t+1} > 0$
  - note that implicitly this takes account of aggregate law of motion
- update ψ by using a weighted average of the new estimate and the old value

Den Haan 1996 algorithm: Advantages & Disadvantages

- even simpler than KS
- even construction of individual policy rules done using simulation methods

Pure perturbation (Roca & Preston)

- KS model has to be modified a little
  - discrete support is likely to be difficult $\Longrightarrow$ continuous support
  - borrowing constraint is definitely difficult $\Longrightarrow$ penalty function
- Perturbation around solution when there is no aggregate and no idiosyncratic uncertainty
- Capital is in levels (not in logs or any other transformation)

Model modifications

- $e_{i,t}$ can take on continuum of values

$$\begin{aligned} e_{i,t+1} &= (1-\rho_e) + \rho_e e_{i,t} + \varepsilon^e_{i,t+1} \\ \varepsilon^e_{i,t+1} &\sim N(0,\sigma^2) \\ \mathsf{E} e_{i,t} &= 1 \Longrightarrow L = 1 \end{aligned}$$

- Continuous penalty term when capital is getting smaller. FOCs:

$$\begin{aligned} c_{i,t}^{-\nu} &= \mathsf{E}_t \left[-2\phi k_{i,t+1}^{-3} + c_{i,t+1}^{-\nu} (r^k_{t+1} + 1 - \delta) \right] \\ k_{i,t+1} &= (1-\delta) k_{i,t} + r_t k_{i,t} + w_t e_{i,t} \bar{l} - c_{i,t} \end{aligned}$$

State variables and order of perturbation

- Individual state variables $s_{i,t} = \{k_{i,t}, e_{i,t}, \tilde{s}_t\}$
- Again a limited set of moments is used as state variables
- As with Xpa, the elements of $\tilde{s}_t$ depend on approximation order
  - First order: $\tilde{s}_t = \{a_t, K_t\}$
  - Second order: $\tilde{s}_t = \{a_t, K_t, \Phi_t, \Psi_t\}$
  -

$$\begin{aligned} K_t &= \int_0^1 (k_{i,t} - K_t)\, di, \\ \Phi_t &= \int_0^1 (k_{i,t} - K_t)^2\, di, \text{ and} \\ \Psi_t &= \int_0^1 (k_{i,t} - K_t)\,(e_{i,t} - \mu_e) di. \end{aligned}$$

What to solve for?

Solve for $h_v(s_{i,t}, \sigma)$ with $v \in \{c, k, K, \Phi, \Psi\}$

What to solve for?

$$\frac{1}{h_c\left(s_{i,t},\sigma\right)} = \beta E_t\left[-2\phi h_k(s_{i,t},\sigma)^{-3} + \frac{(r_{t+1}+1-\delta)}{h_c(s_{i,t+1})}\right]$$

$$h_k(s_{i,t},\sigma) = (1-\delta)k_{i,t} + r_t k_{i,t} + w_t e_{i,t}\bar{l} - h_c(s_{i,t},\sigma)$$

$$K_{t+1} = h_K(\tilde{s}_t,\sigma) = \int_0^1 h_k(s_{i,t},\sigma)di$$

$$\Phi_{t+1} = h_\Phi(\tilde{s}_t,\sigma) = \int_0^1 \left(h_k(s_{i,t},\sigma)-\bar{k}\right)^2 di$$

$$\Psi_{t+1} = h_\Psi(\tilde{s}_t,\sigma) = \int_0^1 \left(h_k(s_{i,t},\sigma)-\bar{k}\right)\left(e_{i,t}-\mu_e\right) di$$

$$r_t = \alpha a_t \left(K_t/\bar{l}\right)^{\alpha-1}$$
$$w_t = (1-\alpha)a_t \left(K_t/\bar{l}\right)^{\alpha}$$

$$s_{i,t+1} = \left\{ \begin{array}{c} h_k(s_{i,t}), 1-\rho_e+\rho_e e_{i,t}+\varepsilon_{i,t+1}, 1-\rho+\rho a_t+\varepsilon_{a,t+1}, \\ h_K(\tilde{s}_t,\sigma), h_\Phi(\tilde{s}_t,\sigma), h_\Psi(\tilde{s}_t,\sigma) \end{array} \right\}$$

- system expressed in period t variables and period $t+1$ shocks
- we now have a perturbation system
  - differentiating system gives values derivatives of h_v
  - from these we get coefficients of Taylor expansions

Comments

- There is no approximation in system specified so far (which is good)
- w_t and r_t depend on mean of the *level* of capital
- We have an exact equation for the mean *because capital is in levels* (and not in logs)
- Capital in logs $\Longrightarrow$ one has to approximate aggregation definition
  - e.g., linearizing around steady state
  - given dispersion in capital levels this may not be accurate

Moments and order of perturbation (first-order)

- Agents obviously care about $k_{i,t}$, $e_{i,t}$, a_t, and K_{t+1}
- First-order perturbation $\Longrightarrow$ linear policy rules $\Longrightarrow$ K_{t+1} only depends on K_t, a_t,and nothing else (mean of $e_{i,t}$ is constant through time)

Moments and order of perturbation (second-order)

- Agents obviously care about $k_{i,t}$, $e_{i,t}$, a_t, and K_{t+1}
- Second-order perturbation $\Longrightarrow$ agent's policy rules depend on $\left(k_{i,t}-\bar{k}\right)^2$ and $\left(k_{i,t}-\bar{k}\right)\left(e_{i,t}-\mu_e\right)$
  - $\Longrightarrow K_{t+1}$ depends on K_t, Φ_t, Ψ_t, and z_t,and nothing else
- Does second-order perturbation include terms like
  - $\left(k_{i,t}-\bar{k}\right)^2 K_t$?
  - $K_t \Psi_t$?
  - $\Psi_t \Phi_t$?

Perturbation combined with Projection

- With idiosyncratic risk you can be quite far from steady state ($\sigma_e = \sigma_a = 0$)
- Idea of Reiter: Focus on steady state implied by $\sigma_a = 0$ and $\sigma_e > 0$
- $\sigma_a = 0 \Longrightarrow$ cross-sectional distribution doesn't change over time $\Longrightarrow$ pretty standard problem to solve

Elements

1. A *numerical* solution to the model:

$$k_{i,t+1} = P_N(e_{i,t}, k_{i,t}, a_t, m_t; \boldsymbol{\lambda_k}),$$

   m_t is a characterization of the distribution

2. Info to be able to generate time series for m_t. E.g.,
   - m_t contains CDF values at a very fine grid
   - distributional assumptions as in AAD

3. #2 $\Longrightarrow P_N(\cdot; \lambda_k)$ pins down aggregate law of motion

$$m_{t+1} = \Gamma_{\lambda_k}(a_{t+1}, a_t, m_t)$$

   That is, λ_k pins down everything

Rewrite the policy function

- Rewrite the *numerical* solution to the model as

$$k_{i,t+1} = P_n(e_{i,t}, k_{i,t}; \boldsymbol{\lambda_{k,t}})$$

with

$$\boldsymbol{\lambda}_{k,\boldsymbol{t}} = \lambda_k(a_t, m_t)$$

- That is, we make λ_k a variable that depends on a_t and m_t

Notation & grid

- $\tilde{s} = [a, m]$
- Dimension of $\boldsymbol{\lambda}_{k,\boldsymbol{t}} = n^{\#}_{\lambda_k}$
- $[\hat{e}, \hat{k}]'$ an $n^{\#}_{\lambda_k} \times 1$ vector with nodes for the employment status and capital levels
    - enough grid points to solve for elements of λ_k

Model equation at grid points

Setting $\delta = 1$ for simplicity

$$\frac{1}{(r(\tilde{s}))\,\hat{k}_j + w(\tilde{s})\hat{e}_j\bar{l} - P_N(\hat{e}_j, \hat{j}_j; \lambda_k(\tilde{s}))} =$$

$$\mathsf{E}\left[\frac{\beta\,(r(\tilde{s}'))}{\left[\begin{array}{c}(r(\tilde{s}'))\,P_N(\hat{e}_j, \hat{k}_j; \lambda_k(\tilde{s})) + w(\hat{s})e'\bar{l} \\ -P_N(e', P_N(\hat{e}_j, \hat{k}_j; \lambda_k(\tilde{s})); \lambda_k(\tilde{s}'))\end{array}\right]}\right]$$

Three more useful equations

$$r(\tilde{s}) = \alpha a \left(K/\bar{l}\right)^{\alpha-1}$$

$$w(\tilde{s}) = (1-\alpha) a \left(K/\bar{l}\right)^{\alpha}$$

Endogenous part of $\tilde{s}'$, i.e., m', is determined by

$$m' = \Gamma_{\lambda_k}(a', a, m),$$

Mental break

- Have I really done anything?
- This system may look more sensible to you without aggregate uncertainty, but so far I have only
  - evaluated first-order conditions at nodes
  - rewritten so that system is a function of current period state variables and next period's realizations

KEY STEPS

- What are the variables in the "new" system?

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- What are the variables in the "new" system?
- How many equations?
- Think of the new system as perturbation system in the variables $\boldsymbol{\lambda}_{k,t}$
  - What is fixed and what are the state variables?

Thus

Write $\lambda_k(\tilde{s})$ as $h_{\lambda_k}(a,m;\Delta^a)$ and its Taylor expansion around the steady state as

$$
\begin{array}{cc}
 & h_{\lambda_k}(a,m;\Delta^a) \\
 & h_{\lambda_k}(\bar{a},\bar{m};0) + h_{\lambda_k,a}(a-\bar{a}) + h_{\lambda_k,m}(m-\bar{m}) + h_{\lambda_k,\Delta^a}\Delta^a \\
= & + \;\; h_{\lambda_k,a}(a-\bar{a})^2/2 + h_{\lambda_k,mm}(m-\overline{m})^2/2 + h_{\lambda_k,\Delta^a\Delta^a}\left(\Delta^a\right)^2/2 \\
 & + \text{ second-order cross products} \\
 & + \cdots
\end{array} .
$$

Dealing with transitions

- KS, Xpa, Den Haan 1996, Roca Preston, & Reiter (hybrid) focus on
    - small changes aggregate variables close to steady state
    - behavior aggregate variables in a *typical* simulation
- This means they cannot deal with
    - transition after a *one-time* and *unforeseen* redistribution of capital
    - destruction of capital

Simple analogy

- Projection solution of the neoclassical growth model gives

$$k' \approx p_N(k, a; \psi)$$

- By using a wide enough grid for k and a and a rich enough approximating function one ensures accuracy for *all* values of k and a inside the grid including those not encountered in a simulation
- In solving heterogeneous agent models can you attain accuracy for any cross-sectional distribution?

Algan, Allais, and Den Haan (2008)

- use projection methods and quadrature techniques as much as possible
  - $\Longrightarrow$ construct a grid for the aggregate state variables including moments
- calculate next-period's moments using quadrature techniques
  - quadrature integration requires functional form for the distribution
  - use flexible functional form and link moments with polynomial's coefficients

AAD - Key step

moments $\iff$ functional form density

AAD - Flexible cross-sectional density

$$P_N(k,\rho) = \rho_0 \exp\left(\begin{array}{c} \rho_1 \left[k - m(1)\right] + \\ \rho_2 \left[(k - m(1))^2 - m(2)\right] + \cdots + \\ \rho_N \left[(k - m(1))^N - m(N)\right] \end{array} \right).$$

where $m(n)$ is the n^{th} uncentered moment

- Goal: find the ρs such that values of moments match implied moments
- This particular functional form $\Longrightarrow$

$$\min_{\rho_1,\rho_2,\cdots,\rho_N} \int_0^\infty P_N(k,\rho)dk.$$

and this is a convex problem

AAD #1: specify aggregate law of motion

- construct grid for aggregate state variables
- calculate next period's moments
- do projection step to calculate aggregate law of motion
- construct grid for individual agent (includes aggregate state variables)
- solve individual problem (for given aggregate law of motion)
- iterate between aggregate and individual problem

AAD #2: do not specify aggregate law of motion

- construct grid for individual problem (includes aggregate state variables)
- calculate next period's mean (and thus r_t and w_t) directly using quadrature techniques
- Solve individual problem

Problem with algorithm so far

- Moments fulfill two roles
  - state variable
  - get the shape of the distribution right
- To get shape right, you need several moments
  - $\Longrightarrow$ you need several state variables
- Solution:
  - use limited set of moments as state variables
  - use additional higher-order moments as reference moments for good shape

Getting the distribution right

- Suppose you only use the mean capital stock as a state variable
- But use $N(K, \sigma_K^2)$ as the cross-sectional distribution
- You still have to find σ_K^2
- AAD get reference moments from a simulation
- reference moments could depend on the state, i.e., $N(K, (\sigma_K(\tilde{s}_t))^2$