Heterogeneous Agents but no Aggregate Uncertainty

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Individual agent

- Subject to i.i.d. idiosyncratic productivity shocks $e_{i,t} \in \{0.9, 1.1\}$
- Incomplete markets
  - only way to save is through holding capital
  - investing small and large amounts "costlier" than intermediate amounts

Individual agent - standard formulation

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t \ln(c_{i,t})$$

s.t.

$$c_{i,t}+k_{i,t}=r_t k_{i,t-1}+w_t e_{i,t}+(1-\delta)k_{i,t-1}$$

$$k_{i,t}\geq 0$$

Individual agent - alternative formulation I

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t \ln(c_{i,t})$$

s.t.

$$c_{i,t} + k_{i,t} = r_{i,t}k_{i,t-1} + w_t e_{i,t} + (1-\delta)k_{i,t-1}$$

$$r_{i,t} = r_t + \frac{\zeta_1}{\zeta_0}\exp(-\zeta_0 k_{i,t})$$

Individual agent - alternative formulation I

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} E \sum_{t=0}^{\infty} \beta^t \ln(c_{i,t})$$

s.t.

$$c_{i,t} + k_{i,t} = r_{i,t} k_{i,t-1} + w_t e_{i,t} + (1-\delta) k_{i,t-1}$$

$$r_{i,t} = r_t + \frac{\zeta_1}{\zeta_0} \exp(-\zeta_0 k_{i,t})$$

Individual agent - alternative formulation II

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t \ln(c_{i,t}) - \frac{\zeta_1}{\zeta_0}\exp(-\zeta_0 k_{i,t})$$

s.t.

$$c_{i,t} + k_{i,t} = r_t k_{i,t-1} + w_t e_{i,t} + (1-\delta)k_{i,t-1}$$

Individual agent - alternative formulation II

- Budget constraint same as original formulation
- Only Euler equation affected
- No non-differentiabilities
- Two ways to think of penalty function:
  - numerical procedure to mimic/implement inequality constraint
  - alternative economic story

Individual agent - Euler equation

$$-\frac{1}{c_{i,t}} + \zeta_1 \exp(-\zeta_0 k_{i,t}) + \mathsf{E}_t \left[\frac{\beta}{c_{i,t+1}} \left(r_{t+1} + 1 - \delta \right) \right] = 0$$

Equilibrium

- Unit mass of worker, $L_t = 1$
- Competitive firm so agent faces competitive prices
  - $w_t = (1-\alpha) K_t^{\alpha}$
  - $r_t = \alpha K_t^{\alpha-1}$
- No aggregate risk so

$$K_t = K$$

- How to find the equilibrium K?

Algorithm

- Guess a value for r
- This implies values for w and K (i.e., demand for capital)
- Solve the individual problem with this value for r (and w)
- Calculate supply of capital K^{supply}
  - e.g., by simulating
- If $K^{\text{supply}} < K$ then r too low so raise r, say

$$r^{\text{new}} = r + \lambda(K - K^{\text{supply}})$$

- Iterate until convergence

Calculating average K

1. First procedure:
    1. Start with a cross-section of many agents
    2. Simulate until cross-sectional average of capital has converged
2. Second (simpler) procedure:
    1. Simulate capital stock for one agent
    2. Take time-series average
        - disregard some initial observations to avoid dependence on weird initial value

Calculating average K without simulating

Consider the simple grid method to simulate

- $p_{e,t}$: vector with fraction of employed agent's at grid points for k
- $p_{u,t}$: vector with fraction of employed agent's at grid points for k

$$p_{t+1} = Ap_t$$

The ergodic distribution p is the normalized eigenvector corresponding to the unit eigenvalue of A

Calculating average K without simulating

$$p_{t+1} = Ap_t$$

Where does this come from?

$$\begin{aligned} p_{e,t+1} &= \lambda_e \Omega_e p_{e,t} + (1-\lambda_u)\Omega_u p_{u,t} \\ p_{u,t+1} &= \lambda_u \Omega_u p_{u,t} + (1-\lambda_e)\Omega_e p_{e,t} \end{aligned}$$

$$p_{t+1} = \begin{bmatrix} \lambda_e \Omega_e & (1-\lambda_u)\Omega_u \\ (1-\lambda_e) & \lambda_u \Omega_u \end{bmatrix} p_t$$

Calculating average K without simulating

$$\Omega_e = \begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\omega_1 & \omega_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\omega_1 & 1-\omega_2+\omega_3 & & & & & & & & \\
 & 1-\omega_3 & & & & & & & &
\end{array}$$

Explicit aggregation - linear case

- Suppose individual policy rule is linear:

$$k_{i,t} = b_0 + b_1 k_{i,t-1} + b_2 \left(e_{i,t} - 1\right)$$

- Implied aggregate policy rule:

$$K_t = b_0 + b_1 K_{i,t-1}$$

- Thus, implied supply of capital:

$$K = b_0 / (1 - b_1)$$

Explicit aggregation - second order

- Suppose individual policy rule is quadratic:

$$k_{i,t} = \begin{array}{c} b_0 + b_1 k_{i,t-1} + b_2 (e_{i,t} - 1) \\ +b_3 k_{i,t-1}^2 + b_4 k_{i,t-1} (e_{i,t} - 1) + b_5 (e_{i,t} - 1)^2 \end{array}$$

- Implied aggregate policy rule:

$$K_t = \begin{array}{c} b_0 + b_1 K_{t-1} + 0 \\ +b_3 M_{t-1}(2) + 0 + b_5 \sigma_e^2 \end{array}$$

where

$$M_{t-1}(2)$$

is the cross-sectional mean of

$$k_{i,t-1}^2$$

Explicit aggregation - second order

- At the ergodic distribution:

$$\begin{aligned} K_t &= K_{t-1} = K \\ M_t(2) &= M_{t-1}(2) = M(2) \end{aligned}$$

- $$K = \frac{b_0 + b_3 M(2) + b_5 \sigma_e^2}{1 - b_1}$$

- How to calculate $M(2)$?

Explicit aggregation - how to calculate M(2)?

$$k_{i,t}^2 = \begin{pmatrix} b_0 + b_1 k_{i,t-1} + b_2 \left(e_{i,t} - 1\right) \\ + b_3 k_{i,t-1}^2 + b_4 k_{i,t-1} \left(e_{i,t} - 1\right) + b_5 \left(e_{i,t} - 1\right)^2 \end{pmatrix}^2$$

- Explicitly aggregating RHS gives

$$M_{t-1}(3) \text{ and } M_{t-1}(4)$$

- Continuing like this will go on forever

Explicit aggregation - how to calculate M(2)?

- Instead of using

$$k_{i,t}^2 = \begin{pmatrix} b_0 + b_1 k_{i,t-1} + b_2 (e_{i,t} - 1) \\ +b_3 k_{i,t-1}^2 + b_4 k_{i,t-1} (e_{i,t} - 1) + b_5 (e_{i,t} - 1)^2 \end{pmatrix}^2$$

use

$$k_{i,t}^2 = \begin{pmatrix} d_0 + d_1 k_{i,t-1} + d_2 (e_{i,t} - 1) \\ +d_3 k_{i,t-1}^2 + d_4 k_{i,t-1} (e_{i,t} - 1) + d_5 (e_{i,t} - 1)^2 \end{pmatrix}$$

- Explicitly aggregating RHS gives

$$M(2)_t = \begin{matrix} d_0 + d_1 K_{t-1} + 0 \\ +d_3 M_{t-1}(2) + 0 + d_5 \sigma_e^2 \end{matrix}$$

Explicit aggregation - how to calculate M(2)?

- At the ergodic distribution

$$K = \begin{array}{c} b_0 + bK + 0 \\ + b_3 M(2) + 0 + b_5 \sigma_e^2 \end{array}$$

$$M(2) = \begin{array}{c} d_0 + d_1 K + 0 \\ + d_3 M(2) + 0 + d_5 \sigma_e^2 \end{array}$$

- That is, linear system in two unknowns

Explicit aggregation - higher order

- If you have n^{th} order solution for $k_{i,t}$ then it is very easy to get n^{th} order solution for $k_{i,t}^{m}$ for any m
- This is the fastest way to solve the model without aggregate uncertainty
  - !!! you don't have to simulate

Explicit aggregation - ???

- Explicit aggregation: calculate average K
  - without simulating and
  - without calculating a distribution
- How is this possible?