Higher-order perturbation & pruning

Wouter J. Den Haan

University of Amsterdam

March 12, 2009

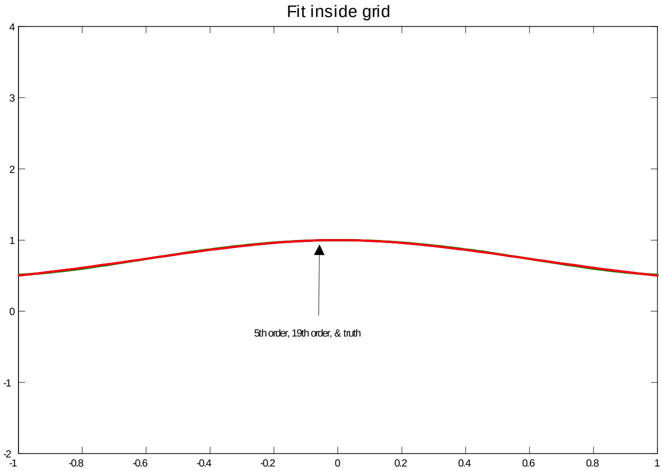
Polynomial approximations

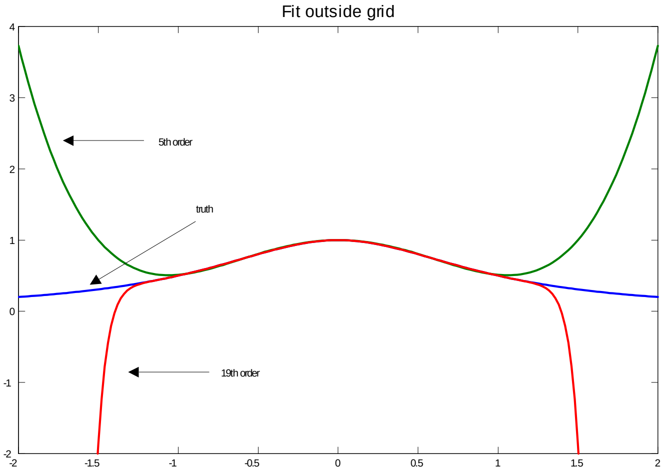
- Polynomial approximations can oscillate
- They always oscillate (relative to the truth) in between grid points
- Hopefully, these oscillations do not affect shape
- Oscillations tend to be *wild* outside the grid
- How is this with higher-order perturbation

Polynomial approximations on a grid

$$f(x) = \frac{1}{1+x^2}$$

- fit on [-1,1] using Chebyshev nodes
- evaluate on [-2,2]


Fit inside grid
5th order, 19th order, & truth
4
3
2
1
0
-1
-2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Fit outside grid
5th order
truth
19th order
4
3
2
1
0
-1
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Simple example with penalty function

$$\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma} - P(a_{t+1})$$

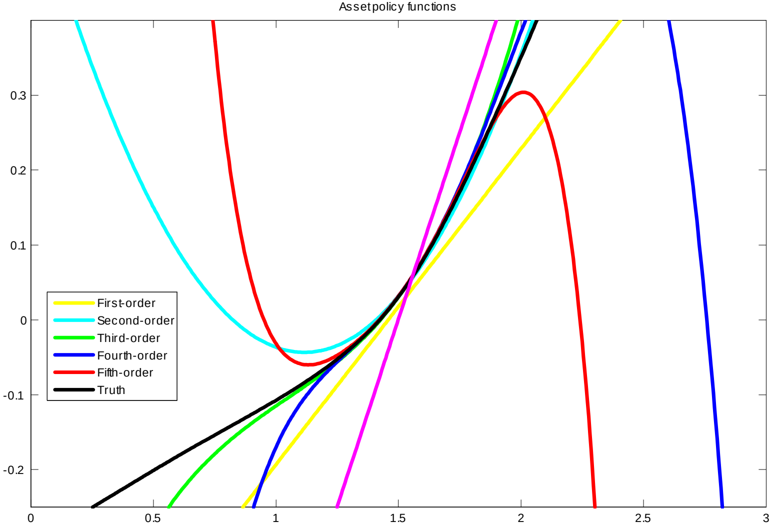
s.t.

$$c_t + a_{t+1}/(1+r) = a_t + y_t = w_t$$

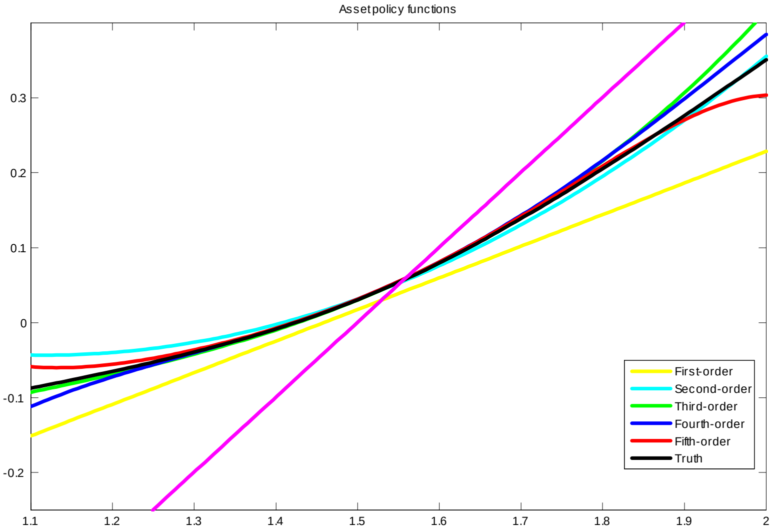
$$P(a_{t+1}) = \frac{\eta_1}{\eta_0} \exp(-\eta_0 a_{t+1}) - \eta_2 a_{t+1}$$

$$y_t = \exp(\mu + \varepsilon_t), \;\; \varepsilon_t \tilde{} N(0, \sigma^2)$$

Policy functions



Policy functions



Adjust model to avoid problem

- Problem here is caused by penalty function
- If you have n^{th}-order approximation, then better to directly specify 2nd-order specification for penalty term showing up in first-order condition

Adjust numerical solution when simulating: pruning

- Proposed by Kim, Kim, Schaumburg, and Sims (2008)
- A *trick* to make sure the simulated series do not explode
- Procedure proposed here is slightly different (see below)

Idea behind pruning

- Solution from i^{th}-order perturbation solution can be written as

$$\tilde{k}_t^{(i)} = \alpha_1^{(i)}\tilde{k}_{t-1}^{(i)} + \gamma_1^{(i)}z_t + f^{(i)}(\tilde{k}_{t-1}^{(i)}, z_t),$$

  where

  - z_t is the exogenous process
  - $f^{(i)}$ contains the higher-order part of the policy function
  - $\tilde{k}_t^{(i)}$ is capital relative to the *stochastic* steady state
  - $\alpha_1^{(i)} = \alpha_1^{(1)}$ and $\gamma_1^{(i)} = \gamma_1^{(1)}$ for second-order perturbation

Simulating using pruning

Steps:

1. Simulate using
   1. the slope coefficients from first-order solution and
   2. constant from i^{th}-order perturbation solution. Thus

$$\tilde{k}_t^{(1*)} = \alpha_1^{(1)} \tilde{k}_{t-1}^{(1*)} + \gamma_1^{(1)} z_t$$

2. Solution from pruned i^{th}-order perturbation solution:

$$\tilde{k}_t^{(i*)} = \alpha_1^{(i)} \tilde{k}_{t-1}^{(i*)} + \gamma_1^{(i)} z_t + f^{(i)}(\tilde{k}_{t-1}^{(1*)}, z_t),$$

Why does it work?

- $\tilde{k}_t^{(1)}$ is by construction non-explosive
- $\tilde{k}_t^{(i)}$ is also non-explosive (unless uncertainty affects the first-order slope coefficients so much that for higher-order solutions the BK conditions are not satisfied
- Except for that unlikely case, the first step can also use

$$\tilde{k}_t^{(1*)} = \alpha_1^{(i)} \tilde{k}_{t-1}^{(1*)} + \gamma_1^{(i)} z_t$$

Stochastic steady state

- KKSS (2008) use

$$\hat{k}_t^{(1)} = \alpha_1^{(1)} \hat{k}_{t-1}^{(1)} + \gamma_1^{(1)} z_t$$

  and substitute this into

$$\hat{k}_t^{(i)} = \delta^{(i)} + \alpha_1^{(i)} \hat{k}_{t-1}^{(i)} + \gamma_1^{(i)} z_t + f^{(i)}(\hat{k}_{t-1}^{(i)}, z_t),$$

  where $\hat{k}_t$ is the deviation from the non-stochastic steady state and $\delta^{(i)}$ a constant.
- But $\hat{k}^{(1)}$ and $\hat{k}^{(i)}$ have different stochastic steady state
- In contrast, $\tilde{k}^{(1)}$ and $\tilde{k}^{(i)}$ have same stochastic steady state, so you are replacing more similar things

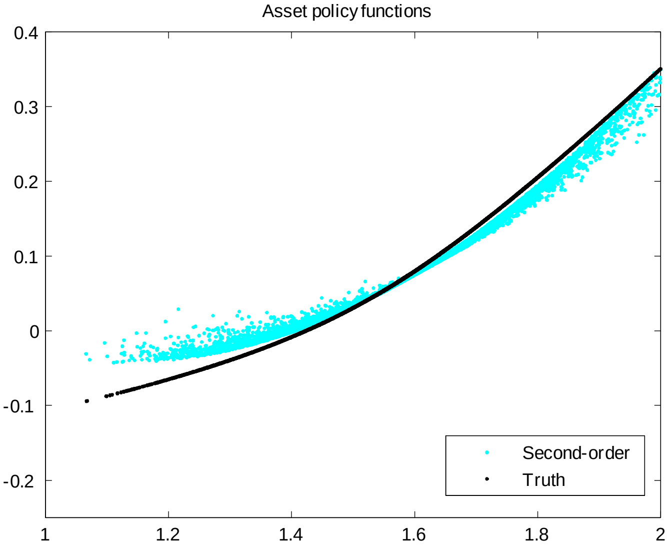
Cost of pruning

- Pruning *distorts* the policy function to ensure stability
- What is the cost?

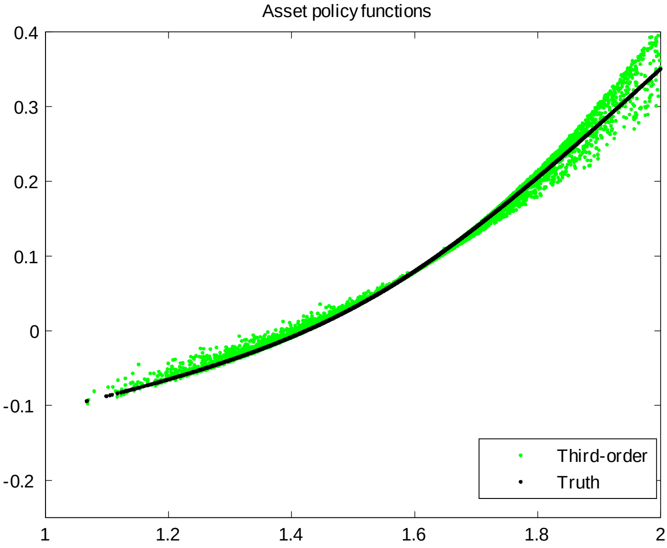
Cost & benefit of pruning in simple example

- second and third-order explode without pruning

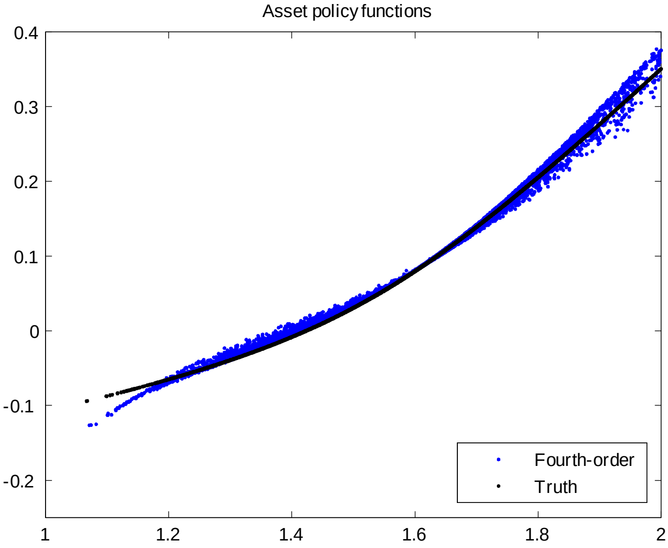
Policy function with pruning - 2nd order



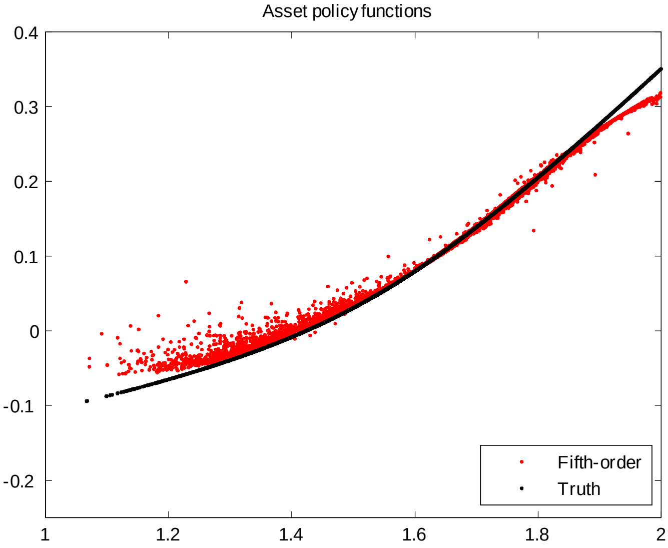
Policy function with pruning - 3rd order



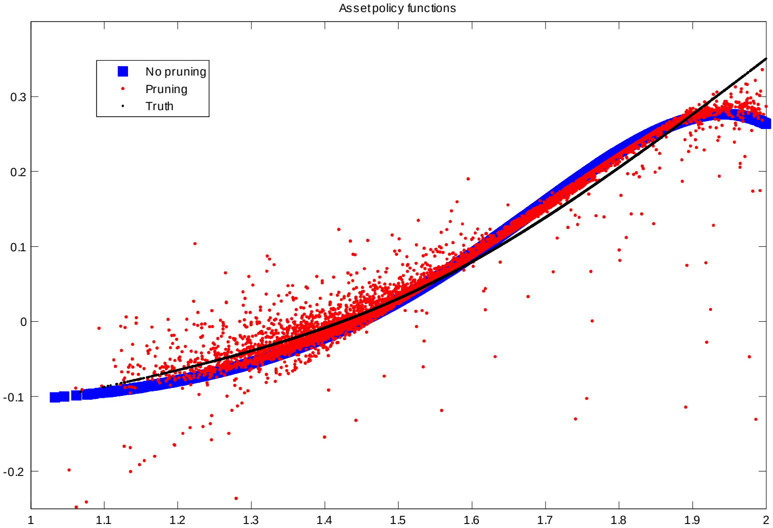
Policy function with pruning - 4th order



Policy function with pruning - 5th order



- It could be worse
- Now with somewhat different parameters



Asset policy functions
No pruning
Pruning
Truth
0.3
0.2
0.1
0
-0.1
-0.2
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2