Dynamic programming and solution methods – 2007/08

Tinbergen Field Course

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1. a Consider the following correspondences defined on the interval X = [0,2]

i) $\Gamma(x) = \{ y \in [5,6] \cup [7,9]\}$

ii) $\Gamma(x) = \{ y \in (5,6] \}$

iii) $\Gamma(x) = \{ y \in [1,6] \}$ for $x \leq 1$

$\Gamma(x) = \{ y \in [1,2] \cup [2,3]\}$ for $x > 1$

Determine which correspondences are uhc and which correspondences are lhc.

i) Is correspondence is lhc because it doesn't suddenly shrink or expand. Moreover, lhc is trivially satisfied. For example, for $y \in \Gamma(x)$ it is trivial to construct a sequence $y_n \in \Gamma(x_n)$. You simply set $y_n = y$. The correspondence is also uhc because $\Gamma(x)$ is bounded and the graph of Γ is closed (Theorem 3.4). The idea is that since the set of *all* feasible points is bounded and closed you will never converge to a point outside this set, that is, you will always converge towards a feasible value.
ii) This correspondence is lhc and the argument is as in part (i). The important thing is to realize that you cannot apply the definition of the book because it is defined for compact-values function. You can find alternative definitions on the web but to be honest I found them to lead to different answers.
iii) This correspondence is uhc and the argument is as in part (i). This correspondence is not lhc. For example, let $x_n = 1 + 1/n$ and let $y = 4$. Clearly you cannot find a sequence $y_n \in \Gamma(x_n)$ such that $y_n \rightarrow y$ since for all values of $x > 1$, $\Gamma(x_n)$ takes on values less than 3 which is pretty far away from 4.

1.b Consider the following correspondences on the set X = [-1,1]

i) $\Gamma(x) = \{ y \in [5,6] \cup [7,9]\}$

ii) $\Gamma(x) = \{ y \in (5,6] \}$

iii) $\Gamma(x) = \{ y \in [0, x^2+5] \}$

iv) $\Gamma(x) = \{ y \in [0, -x^2+5] \}$

Determine which correspondences are convex valued and which correspondences are compact valued. As always explain your answer.

Note that the question asks whether these correspondences are compact and convex valued. To check you have to fix an x and check whether the (one-dimensional) set $\Gamma(x)$ is compact and convex.

i) Since the set is closed and bounded it is compact for all $x \in X$. This set is not convex valued for any x because $y^1 = 6 \in \Gamma(x)$ and $y^2 = 7 \in \Gamma(x)$ (for any $x \in X$) but $y = 0.5*y^1 + 0.5*y^2 = 6.5 \notin \Gamma(x)$.

ii) Again the answer is the same for all values of x. Since there are no gaps in $\Gamma(x)$ the sets are convex and the correspondence is convex valued. Since $\Gamma(x)$ is an open set it is not compact valued.

iii) Recall that you have to condition on x. For every fixed value of x the (one-dimensional) set $\Gamma(x)$ is compact and convex. Thus, it is convex-valued. Note that the set of (x,y) points defined by this correspondence would not be a convex set but that is a different question.

iv) Same as in part iii.

1.c Give an example of a discontinuous function on a compact set that doesn't attain a maximum.

$f(x) = x$ for $x \in [0,5)$ and $f(x) = 0$ for $x \in [5,10]$

2. a What is the sup norm and what is the L^2 norm?

b. Can polynomials approximate the following (continuous but non-differentiable) function arbitrarily well? Does your answer depend on whether you use the sup norm or the L^2 norm?

$f(x) = 0 \quad \text{for } x \leq 0$

$f(x) = x \quad \text{for } x > 0$

This function is continuous so can be approximated arbitrarily well with a polynomial using both the sup and the L^2 norm on a compact interval. You have to realize, however, that this is a very hard function to approximate with a polynomial because of the non-differentiability. Note that by using splines this becomes a trivial function to approximate.

If the interval is not bounded then it becomes a difficult problem. Let the bound depend on the order of the polynomial, that is, the function is defined on $[-B_n,B_n]$. If B_n goes to infinity with n slowly enough then one would think that the standard theorems would go through. But it is a tricky problem especially if--as is the case in this problem--the function value goes to infinity.

3. Discuss the relationship between the sequence of functions g_n:X→X converging uniformly and converging in the sup norm.

Uniform convergence means that $\forall\varepsilon>0, \exists N, \ni$ for $n>N$ $|g_n(x)-g(x)| < \varepsilon\ \forall \boldsymbol{x}\in\boldsymbol{X}$. Key is the last part, $\forall \boldsymbol{x}\in\boldsymbol{X}$. That is, you have to find one value of N such that g_n is close to g for all function values in the domain. But this is implied by convergence in the sup norm, because the sup norm takes the maximum difference across all values of x.